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Duality symmetry and the Cardy limit

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Abstract: We study supersymmetric and non-supersymmetric extremal black holes obtained in Type IIA string theory compactified on $K3\times T^2$, with duality group $O(6,22,\mathbb{Z})\times \mathrm{SL}(2,\mathbb{Z})$. In the Cardy limit an internal circle combines with the AdS_2 component in the near horizon geometry to give a BTZ black hole whose entropy is given by the Cardy formula. We study black holes carrying $D0-D4$ and $D0-D6$ brane charges. We find, both in the supersymmetric and non-supersymmetric cases, that a generic set of charges cannot be brought to the Cardy limit using the duality symmetries. In the non-supersymmetric case, unlike the supersymmetric one, we find that when the charges are large, a small fractional change in them always allows the charges to be taken to the Cardy limit. These results could lead to a microscopic determination of the entropy for extremal non-supersymmetric black holes, including rotating cases like the extreme Kerr black hole in four dimensions.

Keywords: Black Holes in String Theory, String Duality.

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Contents

1. Introduction

Black Holes continue to be a fascinating subject for study in string theory. A central question is to understand the microstates of these black holes and compare their counting with the Bekenstein-Hawking entropy. This was first done for big black holes in 5 dimensions in the classic work of Strominger and Vafa [1]. There have been several important subsequent developments, see for example the reviews, [2 – 5], and references therein. Subleading corrections have been analysed more recently,(see [6 – 10]), [11], and related to the topological string partition function in [12]. For small black holes the pioneering work was done by Sen, (See [13] and [14]) and developed further with precise agreement being found between the microstate counting and the Bekenstein-Hawking-Wald entropy in [15 – 18].

The microscopic descriptions that have been developed so far are usually in terms of a 1+1 dim. Conformal Field Theory (CFT). Furthermore the microscopic counting has been done most reliably in the thermodynamic limit of the CFT, see e.g., [1, 19, 20], and the reviews, [2 – 5], and references therein; some papers which discus the microscopic counting for non-supersymmetric black holes are[1] [24 – 26]. In terms of the energy, L_0, and central charge of the CFT, C, the condition for the thermodynamic limit to be valid takes the form,

$$L_0 \gg C. \tag{1.1}$$

For a supersymmetric or non-supersymmetric extremal black hole, L_0 and C are determined by the charges carried by the black hole. The entropy in this limit is given by the well known Cardy formula,

$$S = 2\pi\sqrt{\frac{CL_0}{6}}. \tag{1.2}$$

In the discussion below, we will often refer to the thermodynamic limit as the Cardy limit. We see from eq. (1.2) that in this limit a knowledge of the central charge and the energy, L_0, is sufficient to determine the entropy. Moreover, the central charge is a robust quantity which can often be determined quite easily by anomaly considerations. This makes it easy to carry out a microscopic calculation of the entropy, [27 – 29].

In addition, when the condition, eq. (1.1) is valid subleading corrections to the entropy can also often be easily calculated. These continue to have the form, eq. (1.2). The subleading corrections arise due to corrections to the central charge, C and can be determined by anomaly considerations [30 – 33].

Since so much can be understood in the Cardy limit, it is natural to ask whether any charge configuration can be put in the Cardy limit using the duality symmetries of string theory. This is the main question we will explore in this paper. Our focus is on big black holes. These carry large charges, $Q \gg 1$, and have a horizon radius which is large compared to the Planck and string scales, so that their horizon geometry is well described by the supergravity approximation. We are interested in both supersymmetric and non-supersymmetric extremal black holes of this type.

We will focus on black holes obtained in Type IIA string theory compactified on $K3 \times T^2$, with duality group, $O(6, 22, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$. For a configuration with $D0 - D4$ brane charges we identify some necessary conditions which must be met. Generically, it turns out that these conditions cannot be met, leading to the conclusion that a generic set of charges cannot be taken to the Cardy limit. These results are valid for both supersymmetric and non-supersymmetric extremal black holes. We find that the required non-genericity, to be able to take a set of charges to the Cardy limit, is interestingly different in the two cases. In the non-supersymmetric case, unlike the supersymmetric one, a "near-by" charge configuration can always be found which can be brought to the Cardy limit. The fractional shift in the charge required to go to the near-by configuration, satisfies the condition,

$$\frac{\Delta Q}{Q} \sim \frac{1}{\sqrt{Q}}, \tag{1.3}$$

[1]For recent developments on rotating black holes, see, [21 – 23]

and is small for large charge. Similar results are also shown to hold in the $D0-D6$ system, which is non-supersymmetric. In this case one can never take the charges to the Cardy limit, but again, a small alteration in the charges brings us to a $D0-D2-D4-D6$ system which can be taken to the Cardy limit. Our results can be extended to some more general charges in a straightforward way. We also expect similar results to hold in other compactifications, for example of Type IIA on T^6, and Heterotic theory on $K3 \times T^2$.

It is important to emphasise that the results mentioned above arise because the duality group is discrete. If instead of $O(6,22,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ we consider the continuous group, $O(6,20,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, then it is well known that it is always possible to bring a configuration with large charges[2] to the Cardy limit. The continuous group has only one invariant, $I = \vec{Q}_e^2\vec{Q}_m^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2$, where, $\vec{Q}_e, \vec{Q}_m$ are the 28 dimensional electric and magnetic charge vectors. Thus any set of charges, $(\vec{Q}_e, \vec{Q}_m)$, can always be transformed to one in the Cardy limit, with the same value of this invariant. The discrete group is smaller and there are additional discrete invariants that characterise its representations. It should be possible to understand the obstruction to bringing a general set of charges to the Cardy limit in terms of these additional invariants and also understand the required non-genericity in terms of these invariants. We leave this more complete analysis for the future.

If the charges lie in the Cardy limit, the black hole admits a description as a BTZ black hole in AdS_3, in some region of moduli space. It can therefore be regarded as a state in a $1+1$ dim. CFT one and its entropy is given by the Cardy formula, eq. (1.2). Our result, that a generic non-supersymmetric state, after a small shift in charges, can be brought to the Cardy limit, thus tells us that at least in some region of moduli space the entropy of the corresponding black hole can be understood microscopically.

This is a promising start but one would like to do better. In fact the long-term goal behind this work is to try and get an understanding of entropy for four- dimensional extremal non-supersymmetric black holes. The near horizon geometry of these black holes is $AdS_2 \times S^2$. In some cases an internal circle combines with the AdS_2 component giving rise to a locally AdS_3 space, but even in these cases generically the charges do not lie in the Cardy limit. What our result shows is that at least in some region of moduli space, the entropy of such a black hole can be understood microscopically. In this region of moduli space the geometry is that of a BTZ black hole in AdS_3 space. We discuss in the conclusions how an argument might be developed with this starting point, leading to a microscopic derivation of the entropy in other regions of moduli space where the black hole is four dimensional. Such an argument should also be applicable to rotating black holes, including the extreme Kerr black hole in four dimensions.

One comment is worth making at this stage.[3] Sometimes the condition eq. (1.1) is not necessary and a much weaker condition suffices. This happens for example in the D1-D5-P system when the CFT is at the orbifold point. At this point in the moduli space the twisted sectors can be thought of as multiply wound strings. In the singly wound sector the relevant condition is given by eq. (1.1). In contrast in the maximally wound sector the

[2]By large charge we mean that both $Q \gg 1$ and $I \gg 1$.

[3]We thank S. Mathur and A. Strominger for emphasising this point to us.

effective central charge is order one and energy is given by replacing L_0 by,

$$L_0 \to L_0 Q_1 Q_5, \tag{1.4}$$

where Q_1, Q_5 are the $D1, D5$ brane charges. Thus the condition, eq. (1.1), is automatically met for large charges in the maximally wound sector.

Away from the orbifold point though the different twisted sectors mix. The only condition which can now guarantee the validity of the Cardy formula is eq. (1.1), which ensures that the system is in the thermodynamic limit. It is well known that the CFT dual to the Black hole is not at the orbifold point. Thus a microscopic calculation of the entropy using the Cardy formula would require this condition to be valid. In the supersymmetric case, where one is calculating an index, one can still justify working at the orbifold point, where the dominant contribution comes from the maximally wound sector, and hence one would not need to impose the condition, eq. (1.1). However, for non-supersymmetric black holes, which are the ones of primary interest in this paper, the entropy can change as one moves in moduli space. A legitimate microscopic calculation in this case would have to be done away from the orbifold point and would require the condition, eq. (1.1), to hold for the Cardy formula to be valid.

It should be mentioned that the mass gap for excitations above the BTZ black hole can be calculated in the gravity side and is well known to go like,

$$E_{\text{gap}} \sim 1/(LC), \tag{1.5}$$

where L is the length of the circle on which the CFT lives. This shows that an effective picture in terms of one multiply wrapped long string must continue to hold even away from the orbifold point. However a first principles argument of why this happens is still missing especially in the non-supersymmetric case. In the absence of such an argument it is appropriate to require, at least in a first principles calculation of the microscopic entropy, that for the Cardy formula to be valid the condition, eq. (1.1), holds. This paper explores how restrictive this condition is, once the duality symmetries of string theory are taken into account.

The paper is organised as follows. We start with some background in §2. In §3, we discuss the $D0-D4$ system, and in §4, the $D0-D6$ system. In §5, we prove that for the lift in M-theory to give a locally AdS_3 space the $D6$-brane charge must vanish. We end with some conclusions in §6. The appendices, A-D, contain supporting results and discussion.

2. Background

The compactification of Type IIA theory on $K3 \times T^2$ preserves 16 supersymmetries. It is dual to Heterotic theory on T^6 [34]. The resulting four dimensional theory has 28 gauge fields. In the IIA description these arise as follows. One gauge field comes from the RR 1-form gauge potential, C_1; 23 gauge fields from the KK reduction of the RR 3-form gauge potential, C_3, on the 22 non-trivial 2-cycles of $K3$ and on the T^2; and 4 gauge fields from the KK reduction of the metric and the 2-form NS field, B_2 on the 1-cycles of the T^2. The duality group is $O(6,22,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. $O(6,22,\mathbb{Z})$ is the T-duality group of the Heterotic theory, and $\mathrm{SL}(2,\mathbb{Z})$ is the S-duality symmetry of the 4 dimensional Heterotic theory.

A general state carries electric and magnetic charges with respect to these gauge fields. The electric charges, $\vec{Q}_e$, and the magnetic charges, $\vec{Q}_m$, take values in a lattice, $\Gamma^{6,22}$, which is even, self-dual and of signature, $(6,22)$. The lattice is invariant under the group, $O(6,22,\mathbb{Z})$. The electric and magnetic charges, $\vec{Q}_e, \vec{Q}_m$, transform as vectors of $O(6,22,\mathbb{Z})$. And together, $(\vec{Q}_e, \vec{Q}_m)$, transform as a doublet of $\mathrm{SL}(2,\mathbb{Z})$. In a particular basis, $\{e_i\}$ of $\Gamma^{6,22}$, the matrix of inner products,

$$\eta_{ij} \equiv (e_i, e_j), \tag{2.1}$$

takes the form,

$$\eta = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{E}_8 \oplus \mathcal{E}_8 \oplus \mathcal{H} \oplus \mathcal{H}. \tag{2.2}$$

Here $\mathcal{H}$, is given by,

$$\mathcal{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

and $\mathcal{E}_8$ is the Cartan matrix of E_8.

In this basis, the electric charge vector has components,

$$\vec{Q}_e = (q_0, -p^1, q_i, n_1, NS_1, n_2, NS_2). \tag{2.4}$$

Here, q_0 is the $D0$-brane charge; p^1 is the charge due to $D4$-branes wrapping $K3$; $q_i, i = 2, \cdots 23$ are the charges due to D2-branes wrapping the 22 2-cycles of $K3$ which we denote as C_i; n_1, n_2 are the momenta along the two 1-cycles of T^2 and NS_1, NS_2 are the charges due to NS_5 branes wrapping $K3 \times S^1$ where S^1 is one of the two 1-cycles of T^2.

And the magnetic charge vector has components,

$$\vec{Q}_m = (q_1, p^0, p^i, w_1, KK_1, w_2, KK_2). \tag{2.5}$$

Here, q_1 is the charge due to $D2$-branes wrapping T^2; p^0 is the $D6$-brane charge; $p^i, i = 2 \cdots 23$, are the charges due to $D4$-branes wrapping the cycle $\tilde{C}_i \times T^2$, where $\tilde{C}_i$ is the 2-cycle on K3 dual to C_i; w_1, w_2 are charges due to the winding modes of the fundamental string along the two 1-cycles of T^2; and KK_1, KK_2 are the KK-monopole charges that arise along the two 1-cycles of the T^2.

Three bilinears in the charges can be defined,

$$\begin{aligned} \vec{Q}_e^2 &\equiv (\vec{Q}_e, \vec{Q}_e) \\ \vec{Q}_m^2 &\equiv (\vec{Q}_m, \vec{Q}_m) \\ \vec{Q}_e \cdot \vec{Q}_m &\equiv (\vec{Q}_e, \vec{Q}_m). \end{aligned} \tag{2.6}$$

These are invariant under $O(6,22,\mathbb{Z})$.

An invariant under the full duality group is,

$$I = (\vec{Q}_e)^2 (\vec{Q}_m)^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2. \tag{2.7}$$

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It is quartic in the charges. For a big supersymmetric black hole, I is positive, and the entropy of the black hole [35] is,

$$S = \pi\sqrt{\vec{Q}_e^2\vec{Q}_m^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2}. \tag{2.8}$$

In contrast, for a big non-supersymmetric extremal black hole, I is negative and the entropy is,

$$S = \pi\sqrt{(\vec{Q}_e \cdot \vec{Q}_m)^2 - \vec{Q}_e^2\vec{Q}_m^2}. \tag{2.9}$$

We now turn to discussing the Cardy limit. Consider a Black hole carrying $D0-D4$ brane charge. In our notation the non-zero charges are, $q_0, p^1, p^i, i = 2, \cdots 23$. This solution can be lifted to M-theory, and the near horizon geometry in M-theory is given by a BTZ black hole in $AdS_3 \times S^2$. The AdS_3 space-time admits a dual description in terms of a $1+1$ dim. CFT living on its boundary. The central charge, C, of the CFT can be calculated from the bulk, it is determined by the curvature of the AdS_3 spacetime. For large charges we get,

$$C = 3|p^1 d_{ij} p^i p^j|, \tag{2.10}$$

where d_{ij} is the matrix η_{ij}, eq. (2.1), restricted to the 22 dimensional subspace of charges given by $D4$-branes wrapping two-cycles of $K3$ and T^2. This corresponds to the second, third and fourth factor of $\mathcal{H}$ and the two $\mathcal{E}_8$'s in eq. (2.2).

The BTZ black hole is a quotient of AdS_3 obtained by identifying points separated by a space-like direction. The symmetry of AdS_3 is $\mathrm{SO}(2,2)$; this is broken by the identification of points in the BTZ black hole to $\mathrm{SO}(2,1) \times \mathrm{U}(1)$. The size of the circle obtained by this identification, L, is given in terms of the radius of AdS_3, R_{AdS}, by

$$\frac{L}{R_{\mathrm{AdS}}} \sim \frac{|q_0|}{C}, \tag{2.11}$$

where q_0 is the zero-brane charge carried by the Black hole.

In the Cardy limit the condition,

$$|q_0| \gg C, \tag{2.12}$$

is satisfied. From eq. (2.11) we see that this leads to the condition, $\frac{L}{R_{\mathrm{AdS}}} \gg 1$. From, eq. (2.10) we see for this limit to be valid, the condition,

$$|q_0| \gg |p^1 d_{ij} p^i p^j|, \tag{2.13}$$

must hold. Since, $\frac{L}{R_{\mathrm{AdS}}} \gg 1$, in the Cardy limit, the distance between points which are identified in the BTZ background is much bigger than R_{AdS}. As a result, the effect of the reduced symmetry in the BTZ background, due to taking the quotient, can be neglected in the Cardy limit. The partition function in the bulk can then be calculated using the full symmetries of AdS_3. The resulting answer is the well known Cardy formula,

$$S = 2\pi\sqrt{\frac{C|q_0|}{6}}. \tag{2.14}$$

The Cardy limit corresponds to the thermodynamic limit of the microscopic $1+1$ dim. CFT. In this limit the dimensionless temperature T of the CFT satisfies the condition,

$$T \gg 1. \tag{2.15}$$

Away from the Cardy limit the breaking of SO(2, 2) to SO(2, 1) becomes important and there is no way to calculate the partition function or entropy without knowing more details of the bulk, or the dual boundary conformal field theory.

So far we have considered a system with $D0-D4$ brane charge. What about including other charges? If a $D6$-brane charge is also present, we show in §5, that on lifting to M-theory one does not get an AdS_3 space-time. All other charges are allowed by the requirement that the M-theory lift gives an AdS_3 spacetime in the near-horizon limit. So a general configuration which admits an AdS_3 lift can also include $D2$-brane charges, and non-zero values for $n_1, n_2, w_1, w_2, NS_1, NS_2, KK_1, KK_2$, besides having $D0-D4$ brane charges. The resulting central charge of the $1+1$ dim. CFT after lifting to M-theory is[4]

$$C = 3|p^1 \vec{Q}_m^2|. \tag{2.16}$$

In the more general case, the condition for the Cardy limit is,

$$|\hat{q}_0| \gg C. \tag{2.17}$$

Where, $|\hat{q}_0|$ is,

$$|\hat{q}_0| = \frac{|\vec{Q}_e^2 \vec{Q}_m^2 - (\vec{Q}_e \cdot \vec{Q}_m)^2|}{2|p^1 \vec{Q}_m^2|}. \tag{2.18}$$

Using eq. (2.7), eq. (2.16) and eq. (2.18) this can be written in the form,

$$I \gg 6(p^1)^2(\vec{Q}_m^2)^2. \tag{2.19}$$

To summarise, for a charge configuration to be in the Cardy limit, two conditions must hold. First the $D6$-brane charge, p^0, must vanish. Second, eq. (2.17) or equivalently, eq. (2.19), must be valid. We refer to these two conditions as the Cardy conditions below.

Before proceeding let us note that we are neglecting $1/Q$ corrections in the formula for the central charge, eq. (2.16). For these to be small, the BTZ black hole should be a state in a weakly coupled AdS_3 background. The Radius of the AdS_3 space, R_{AdS}, in units of the three dimensional Planck scale, $l_{\text{Pl}}^{(3)}$, is given by,

$$\frac{R_{\text{AdS}}}{l_{\text{Pl}}^{(3)}} \sim C. \tag{2.20}$$

[4]The central charge is determined by all the branes which are extended strings in the AdS_3. One can see from eq. (2.4), eq. (2.5), that this formula gives a dependence on all of them. Localised excitations, like momentum modes or wrapped 2-branes, correspond to states and do not change the central charge.

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For the BTZ black hole to be a state in a weakly coupled AdS_3 spacetime, $\frac{R_{\rm AdS}}{l_{\rm Pl}^{(3)}} \gg 1$, yielding the condition,[5]

$$C \gg 1. \tag{2.21}$$

The conditions on the charges for the Cardy limit are not duality invariant. This raises the question, when can a charge configuration be brought to the Cardy limit after a duality transformation? This is the central question we address in this paper. In §3 we first address this question for the case where the starting configuration, has $D0 - D4$ brane charges. Our analysis includes both the supersymmetric and non-supersymmetric cases. Following this in §4, we address this question when the starting configuration carries $D0 - D6$ brane charges.

There is one potentially confusing point that we would like to address before going further. In asking whether a system of charges can be brought to the Cardy limit, we are really asking whether any of the internal circles of the compactification can combine with the AdS_2 component of the near horizon geometry and give rise to a three-dimensional BTZ black hole and whether this black hole has charges which lie in the Cardy limit. There are six internal circles for example in the Heterotic description, corresponding to the 6 Hyperbolic lattices, $\mathcal{H}$ in eq. (2.2), and we allow for the internal circle to be any one of them. Our results, mentioned in the introduction, which say that generically this is not possible, mean that for generic charges there is no internal circle which can combine in this manner, yielding the Cardy limit.

There are two ways to carry out the analysis. We can keep the charges fixed and ask whether a suitable circle can be found. This corresponds to a passive transformation, under which the charges are kept fixed but the basis in the charge lattice, with respect to which the components were written in eq. (2.4), eq. (2.5), is changed. Alternatively, we can keep the basis fixed and change the charges, and ask whether the transformed charges meet the required conditions. This corresponds to an active transformation. We will adopt this latter active of point of view in the paper. In this point of view the internal circle which combines and potentially gives rise to a BTZ black hole is kept fixed and in our conventions is the M-theory circle in the IIA description.

3. The $D0 - D4$ system

In this section we analyse the $D0-D4$ system. Subsection 3.1 discusses the supersymmetric case, and subsections 3.2, 3.3, discuss the non-supersymmetric case. In both cases we find that a generic set of charges cannot be brought to the Cardy limit. Subsection 3.4, discuss what happens if starting with generic charges we now allow the charges to vary. We find that in the non-supersymmetric case a near-by charge configuration can always be found

[5]The stronger conditions are, $\frac{R_{\rm AdS}}{l_{11}} \gg 1$, $\frac{R_{S2}}{l_{11}} \gg 1$, and $\frac{V_6}{l_{11}^6} \gg 1$, where R_{S^2}, V_6 are the Radius of the S^2 and volume of the internal space respectively. From these, and the relation, $l_{\rm Pl}^{(3)} = \frac{l_{11}^9}{R_{S^2}^2 V_6}$, the condition, $\frac{R_{\rm AdS}}{l_{\rm Pl}^{(3)}} \gg 1$, follows.

which can be brought to the Cardy limit. Additional relevant material is in appendices A and B.

Our starting configuration for the $D0-D4$ case has non-zero values for q^0, p^1, p^i, in the notation of eq. (2.4), eq. (2.5), and all other charge are vanishing. It is easy to see from eq. (2.6) that

$$\vec{Q}_e \cdot \vec{Q}_m = 0, \tag{3.1}$$

in this case.

In our analysis we are interested in the case of large charges, $|q_0|, |p^1|, |p^i| \gg 1$. The Cardy condition for the starting configuration takes the form, eq. (2.13). We see that for a generic set of initial charges this condition will not be met. Generically all charges will be roughly comparable, $|q_0| \sim |p^1| \sim |p^i| \sim Q \gg 1$ Now the l.h.s. of eq. (2.13) is linear in Q while the r.h.s. is cubic in Q, so generically, for $Q \gg 1$, the inequality, eq. (2.13), will not be met.

Below we formulate a set of necessary condition which must be met, for the final configuration to be in the Cardy limit. For generic initial charges, we find that these conditions are not met. And so we learn that generically a system with $D0-D4$ charge cannot be brought to the Cardy limit. In some special, non-generic cases, these necessary conditions are met. We construct some examples of this type and explicitly find a duality transformation bringing them to the Cardy limit.[6]

Let us denote the final configuration which is obtained after carrying out a duality transformation on the initial $D0-D4$ charges by $(\vec{Q}'_e, \vec{Q}'_m)$. As was pointed out above, the $D6$-brane charge, $p^{0'}$, in the final configuration must vanish for this to happen, and eq. (2.19) must be met.

We can restate eq. (2.19) in the slightly weaker form as,

$$|I| \gg (p^{1'}(\vec{Q}'_m)^2)^2. \tag{3.2}$$

This gives rise to the condition,

$$\left|\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}}\right| \ll \frac{1}{|p^{1'}|}. \tag{3.3}$$

Since $|p^{1'}| > 1$ eq. (3.3) leads to the condition,

$$\left|\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}}\right| \ll 1. \tag{3.4}$$

The final configuration, $(\vec{Q}'_e, \vec{Q}'_m)$ is obtained from the initial one, by the action of a combined $\mathrm{SL}(2,\mathbb{Z})$ transformation and an $O(6,22,\mathbb{Z})$ transformation. Denote the element of $\mathrm{SL}(2,\mathbb{Z})$ by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.5}$$

[6]Of course a trivial way in which this could happen is if the initial configuration, while being non-generic, is itself in the Cardy limit, and meets condition, eq. (2.13). In the example we construct, the initial charges while being rather special are not in the Cardy limit. We find explicitly the duality transformation bringing them to this limit.

By definition, $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$. The $\mathrm{SL}(2,\mathbb{Z})$ transformation acts on the charges as follows,

$$\begin{aligned} \vec{Q}_e &\to a\vec{Q}_e + b\vec{Q}_m \\ \vec{Q}_m &\to c\vec{Q}_e + d\vec{Q}_m. \end{aligned} \tag{3.6}$$

The $O(6,22)$ transformation does not change the value of the bilinears, eq. (2.6), also the initial charges satisfy the condition, $\vec{Q}_e \cdot \vec{Q}_m = 0$. This leads to,

$$(\vec{Q}'_m)^2 = c^2\vec{Q}_e^2 + d^2\vec{Q}_m^2. \tag{3.7}$$

Using eq. (3.4), now gives,

$$\left| c^2\frac{\vec{Q}_e^2}{\sqrt{|I|}} + d^2\frac{\vec{Q}_m^2}{\sqrt{|I|}} \right| \ll 1. \tag{3.8}$$

This condition will play an important role in the discussion below.

3.1 The supersymmetric case

Since eq. (3.1) is true for the $D0 - D4$ system, it follows from eq. (2.7) that the duality invariant, I, is,

$$I = \vec{Q}_e^2\vec{Q}_m^2. \tag{3.9}$$

For a supersymmetric system, $I > 0$, so we see that $\vec{Q}_e^2, \vec{Q}_m^2$ have the same sign. From, eq. (3.7) it follows that $(\vec{Q}'_m)^2$ must also have the same sign as $\vec{Q}_e^2, \vec{Q}_m^2$.

Thus eq. (3.4) takes the form,

$$c^2\frac{|\vec{Q}_e^2|}{\sqrt{I}} + d^2\frac{|\vec{Q}_m^2|}{\sqrt{I}} \ll 1. \tag{3.10}$$

Now by doing an $\mathrm{SL}(2,\mathbb{Z})$ transformation if necessary we can always take the initial charges to satisfy the condition,

$$\left| \frac{\vec{Q}_e^2}{\vec{Q}_m^2} \right| \geq 1. \tag{3.11}$$

(Either this condition is already met or we do the $\mathrm{SL}(2,\mathbb{Z})$ transformation $(\vec{Q}_e, \vec{Q}_m) \to (-\vec{Q}_m, \vec{Q}_e)$ after which it is true).

Using the expression for I in eq. (3.9), eq. (3.11) leads to,

$$\frac{|\vec{Q}_e^2|}{\sqrt{I}} \geq 1. \tag{3.12}$$

Now since c, d are integers, we see that the only way, eq. (3.10) can be met is if, $c = 0$. The resulting $\mathrm{SL}(2,\mathbb{Z})$ matrix must then take the form,

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

From eq. (3.7) it now follows that,

$$(\vec{Q}'_m)^2 = \vec{Q}_m^2. \tag{3.14}$$

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The condition, eq. (3.4), using eq. (3.9), eq. (3.14) then leads to,

$$|\vec{Q}_e^2| \gg |\vec{Q}_m^2| \tag{3.15}$$

A few points are now worth making. Eq. (3.15) is a necessary condition on the initial charges $(\vec{Q}_e, \vec{Q}_m)$ which must be met, to be able to go to the Cardy limit. It is easy to see that this condition will not be met generically. If all the initial charges, q_0, p^1, p^i are of the same order, $Q \gg 1$, then, $\vec{Q}_m^2 = 2d_{ij}p^ip^j$ and $\vec{Q}_e^2 = -2q_0p^1$ are both quadratic in Q and will generically be roughly comparable, so that eq. (3.15) is not met. On the other hand this condition is somewhat less non-generic than the condition required for the initial configuration to be in the Cardy limit, since both sides of the inequality scale like Q^2 in eq. (3.15), while in eq. (2.13) the rhs scales relative to the lhs by a factor of Q^2. Thus one can find initial charges which are not in the Cardy limit, but which meet the condition eq. (3.15). We will present some explicit examples below and show that they can be sometimes brought to the Cardy limit by duality transformations.

Before doing so let us comment that the eq. (3.15) can in fact be somewhat tightened. Let $\gcd(\vec{Q}_e)$ stand for the greatest common divisor of all the integer charges in $\vec{Q}_e$. Then the stronger form of this condition is,

$$|\vec{Q}_e^2| \gg (\mathrm{gcd}\vec{Q}_e)^2|\vec{Q}_m^2| \tag{3.16}$$

In appendix A, we discuss how eq. (3.16) can be derived.

In the example we present next, the starting configuration is not in the Cardy limit, but condition, eq. (3.16) is met. We will present the explicit duality transformation that brings this configuration to the Cardy limit.

3.1.1 An explicit example

We start with the charges,

$$\vec{Q}_e = (-p^1+1, -p^1, 0, 0, 0, 0, 0, \cdots, 0) \tag{3.17}$$
$$\vec{Q}_m = (0, 0, p^2, p^2, 0, 0, 0, \cdots, 0) \tag{3.18}$$

with,

$$(p^1)^2 \gg 3(p^2)^2 \gg 1. \tag{3.19}$$

The quadratic bilinears, eq. (2.6), take the values,

$$\begin{aligned} \vec{Q}_e^2 &= 2(p^1-1)p^1 \\ \vec{Q}_m^2 &= 2(p^2)^2 \\ \vec{Q}_e \cdot \vec{Q}_m &= 0 \end{aligned} \tag{3.20}$$

The invariant, I, eq. (2.7), takes the value,

$$I = 4p^1(p^1-1)(p^2)^2 \tag{3.21}$$

Note that this starting configuration is not in the Cardy limit as these charges do not satisfy the condition, eq. (2.19). But the starting configuration does satisfy eq. (3.16) since, $\gcd(\vec{Q}_e) = \gcd(p, p-1) = 1$, and eq. (3.19) holds.

Now we carry out the transformation, $B \in O(3,3,\mathbb{Z}) \subset O(6,22,\mathbb{Z})$, given by,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} . \tag{3.22}$$

B acts non-trivially on the 6 dimensional sublattice of $\Gamma^{6,22}$, with an inner product given by first three $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ factors in eq. (2.1), and acts trivially on the rest of the lattice. The resulting charges are given by,

$$\vec{Q}_e' = (-p^1, 1, p^1, 0, p^1, p^1, 0, \cdots, 0) \tag{3.23}$$
$$\vec{Q}_m' = (0, 0, p^2, p^2, 0, -p^2, 0, \cdots 0) \tag{3.24}$$

Since the second entry in $\vec{Q}_m'$ vanishes, there is no $D6$-brane charge. From, eq. (3.23) we see that $p^{1'} = -1$. Also,

$$\vec{Q}_m'^{\,2} = 2(p^2)^2. \tag{3.25}$$

Now the Cardy condition requires that,

$$I \gg 6 \left(p^{1'} \vec{Q}_m'^{\,2} \right)^2 . \tag{3.26}$$

Using eq. (3.21), eq. (3.25) and eq. (3.19), we see that this condition is indeed met.

An example where all the final charges are much bigger than unity can be obtained by scaling all the charges above, by $\lambda \gg 1$ and taking

$$(p^1)^2 \gg 3(\lambda)^2 (p^2)^2. \tag{3.27}$$

3.2 The non-supersymmetric case

In the Non-supersymmetric $D0 - D4$ system, I also takes the form, eq. (3.9). By doing an SL$(2,\mathbb{Z})$ transformation if necessary we can assume, without loss of generality that

$$\frac{|\vec{Q}_m^2|}{|\vec{Q}_e^2|} \leq 1. \tag{3.28}$$

For subsequent discussion it is useful to define the parameter, α, as follows,

$$\alpha = \frac{|\vec{Q}_m^2|}{\sqrt{|I|}} = \frac{\sqrt{|I|}}{|\vec{Q}_e^2|} = \sqrt{\frac{|\vec{Q}_m^2|}{|\vec{Q}_e^2|}}. \tag{3.29}$$

where the last two equalities follows from eq. (3.9). We see from eq. (3.28) that

$$\alpha \leq 1. \tag{3.30}$$

Since I is negative, we learn from eq. (3.9) that $\vec{Q}_e^2, \vec{Q}_m^2$ must have opposite signs. There are then two possibilities, either $\vec{Q'}_m^{\,2}$ has the same sign as $\vec{Q}_e^2$, or it has the opposite sign as $\vec{Q}_e^2$. In both cases, eq. (3.8) takes the form,

$$0 < \left| -d^2\alpha + \frac{c^2}{\alpha} \right| \ll 1. \tag{3.31}$$

The requirement $|-d^2\alpha + \frac{c^2}{\alpha}| > 0$ arises from the condition that $\vec{Q'}_m^{\,2}$ is non-vanishing, and this in turn arises from the requirement that the central charge, C, eq. (2.16), does not vanish.

The analysis and conclusions are similar in the two cases. Below we give details for the case when $\vec{Q'}_m^{\,2}$ and $\vec{Q}_e^2$ have the same sign and also state the conclusions for the case when $\vec{Q}_m'^2$ and $\vec{Q}_e^2$ have the opposite sign.

In the case when $\vec{Q}_m'^2, \vec{Q}_e^2$, have the same sign, eq. (3.31) takes the form,

$$0 < -d^2\alpha + \frac{c^2}{\alpha} \ll 1. \tag{3.32}$$

It is interesting to compare this with the condition that arose in the susy case, eq. (3.10). This constraint required the charges to be non-generic and to satisfy the condition, eq. (3.15), in the susy case. In terms of α, defined in eq. (3.29), this condition takes the form,

$$\alpha^2 \ll 1. \tag{3.33}$$

At first sight it might seem that the difference in relative sign between the two terms makes eq. (3.32) easier to satisfy in the non-susy case. To explore this question we will take, $\alpha < 1$, but not much less than unity and ask whether such a set of charges can be brought to the Cardy limit. We will find that in fact eq. (3.32) cannot be met for generic initial charges. Also, we will see that the nature of the non-genericity which allows eq. (3.32) to be met is interestingly different from the susy case, and this has interesting consequences which we will discuss further in the next subsection.

Conditions, eq. (3.30) and eq. (3.31), and the fact that c takes integer values, imply that d cannot vanish. We can then write eq. (3.32) as follows,

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \ll 1. \tag{3.34}$$

Since $d^2 \geq 1$ and $\alpha \leq 1$, this gives rise to a weaker condition,

$$0 < \left(-\alpha + \left|\frac{c}{d}\right|\right)\left(\alpha + \left|\frac{c}{d}\right|\right) \ll 1. \tag{3.35}$$

Now if α is not very much less than unity, as we are assuming, then $(\alpha + |\frac{c}{d}|)$ cannot be very much less than unity. Thus the only way to meet the condition, eq. (3.35), is for

$$0 < \left|\frac{c}{d}\right| - \alpha \ll 1. \tag{3.36}$$

In general we see from eq. (3.29) that α is an irrational number and $|\frac{c}{d}|$ is a rational number. We know that any irrational number can be approximated arbitrarily well by a rational number, therefore one can meet condition eq. (3.36) for a general α.

Let us however go back to the stronger condition, eq. (3.34), we will see that this cannot be met generically. We state the condition in eq. (3.34) as follows:

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) < \delta, \tag{3.37}$$

where, δ is a small number satisfying,

$$\delta \ll 1. \tag{3.38}$$

Eq. (3.36) then takes the form,

$$0 < \left|\frac{c}{d}\right| - \alpha < \delta. \tag{3.39}$$

As was mentioned above, since any irrational number can be approximated arbitrarily well by a rational number, c, d can always be found so that eq. (3.39) is met. However, for a generic irrational number, α, the integers, d, c that satisfy eq. (3.39) will have to be of order $O(1/\delta)$.[7] Approximating,

$$\alpha + \left|\frac{c}{d}\right| \sim 2\alpha, \tag{3.40}$$

we see that

$$\frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \simeq 2d^2\left(-\alpha + |\frac{c}{d}|\right) \sim O(1/\delta). \tag{3.41}$$

It then follows that eq. (3.34) will not be generically met, since δ satisfies the condition, eq. (3.38).

In other words, while α can be approximated arbitrarily well by the ratio of two integers, $|c/d|$, in general doing so to better accuracy by choosing δ to be smaller will make the condition, eq. (3.34), harder to meet.

The condition in eq. (3.34) can be met if α is a non-generic irrational number for which eq. (3.39) can be met by taking

$$c, d \sim O\left(\frac{1}{\delta^{1/2-\epsilon}}\right). \tag{3.42}$$

with $\epsilon > 0$. In this case one finds that,

$$\frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \sim O(\delta^{2\epsilon}), \tag{3.43}$$

and thus eq. (3.34) can be met if $\delta \ll 1$.

An example is provided by

$$\alpha = \sqrt{\frac{p-1}{p}}. \tag{3.44}$$

[7]For example to approximate $1/\sqrt{2} = 0.707106\ldots$, to n significant figures, c, d would have to be $O(n)$.

It is easy to see that eq. (3.34) is met in this case if $c = d = 1$ and $p \gg 1$. This example, fits in with the discussion above. The irrational number α, in this case, is well approximated to $O(1/p)$ by two integers which are unity, and which therefore satisfies the condition, eq. (3.42).

The example above can be easily generalised to the case,

$$\alpha = \frac{m}{n}\sqrt{\frac{p-1}{p}} \tag{3.45}$$

where $m < n$ and $mn \ll p$. Once again eq. (3.34) can be met, by taking, $c = m, d = n$. We will have more to say about what these examples are teaching us in the following subsection, where we consider varying the charges.

To summarise the discussion above, we have learned that eq. (3.34) can be met, but only for rather special values of the initial charges. These charges are such that α is of the form,

$$\alpha = \frac{m}{n} - \epsilon, \tag{3.46}$$

where $0 < \epsilon \ll 1$, and the integers, m, n are not very big, and meet the condition,

$$2n^2\epsilon \ll 1. \tag{3.47}$$

In this case, by taking, $c = m, d = n$ eq. (3.34) can be met.[8]

There is another way to characterise the non-genericity of α. Suppose we choose the initial charges such that α took a special value, eq. (3.46), and integers, c, d exist meeting conditions, eq. (3.34). We could ask by how much can the initial charges be varied so that integers c, d continue to exist, meeting the condition eq. (3.37). If all the initial charges are of order Q and they are varied by a small amount ΔQ, we have that,

$$\frac{\Delta\alpha}{\alpha} \sim \frac{\Delta Q}{Q}. \tag{3.48}$$

Using, eq. (3.40), we can write the condition, eq. (3.37) as,

$$0 < \frac{d^2}{\alpha}\left(-\alpha^2 + \frac{c^2}{d^2}\right) \simeq 2d^2\left(-\alpha + \frac{c}{d}\right) < \delta. \tag{3.49}$$

Now, when

$$\Delta\alpha \sim \frac{\delta}{2d^2}, \tag{3.50}$$

c, d will have to change from their initial values for, the inequality, eq. (3.49) to continue to hold. But for a generic small variation, new integers, c, d, cannot be found meeting condition, eq. (3.42), rather the new integers will be of order $O(1/\delta)$ and as a result eq. (3.37) will not be met. Therefore the maximum variation for the initial charges is of order,

$$\frac{\Delta Q}{Q} \sim \frac{\delta}{2d^2}. \tag{3.51}$$

[8]For the matrix eq. (3.5) to exist c, d must be coprime. This requires that we cancel off any common factors in m, n and take them to be coprime.

Since δ satisfies eq. (3.38), and d is a non-vanishing integer, we see that this variation is small.

To summarise, in this subsection we have seen that a non-supersymmetric system carrying generic $D0-D4$ brane charges cannot be brought to the Cardy limit after a duality transformation. The case when α is rational needs to be treated somewhat differently, we analyse this case below. Some examples, of non-generic charges, which can be brought to the Cardy limit using the duality symmetry are discussed in appendix B.

3.3 Rational α

Since we saw that α had to be close to a rational number for the integers c, d to exist meeting the condition in eq. (3.34), it might seem at first that for any α which is rational one can always meet this condition. We show here that this is not true, eq. (3.34) can be met by rational α but again of a rather special form.

Suppose that

$$\alpha = \frac{m}{n} \tag{3.52}$$

so that ϵ in eq. (3.46) vanishes. We will again take the case where $\alpha < 1, \alpha \not\ll 1$.[9] Without loss of generality, we can take m, n to be co-prime. One could now choose $d = m, c = n$ so that

$$\left|\frac{d}{c}\right| - \alpha = 0 \ll 1. \tag{3.53}$$

However in this case we see that eq. (3.34) is not met at the other end, since, $(|\frac{d}{c}| - \alpha) \not> 0$.

We need to find integers, c, d such that $|\frac{d}{c}|$ is close to α, but does not exactly cancel it. This will not be generically possible for exactly the same reason as the case of irrational α. To meet the condition eq. (3.39), c, d will generically be of order $1/\delta$, while to meet eq. (3.34) they would need to meet condition eq. (3.42). These two requirements are not compatible.

To understand when the condition in eq. (3.34) can be met more precisely, let us write this equation as,

$$0 < \frac{1}{\alpha}(\alpha|d| + |c|)(-\alpha|d| + |c|) \ll 1. \tag{3.54}$$

Now since, $|c| > \alpha|d|$ we have, $|c| + |d|\alpha > 2|d|\alpha$, and it follows from eq. (3.54) that,

$$0 < \frac{2|d|}{n}(n|c| - m|d|) \ll 1. \tag{3.55}$$

Since, the minimum non-vanishing value of $(n|c| - m|d|)$ is unity, one consequence of eq. (3.55) is that, $n/|d| \gg 1$. Given that α is not much smaller than unity it follows then that,

$$m, n \gg 1. \tag{3.56}$$

Also since, $2|d| > 1$, it follows from eq. (3.55) that

$$0 < \frac{n|c| - m|d|}{n} \ll 1. \tag{3.57}$$

[9]We impose this restriction since if $\alpha \ll 1$, the charges are be non-generic to start with.

In summary, if α is a rational number, $\alpha = m/n$, an $\mathrm{SL}(2,\mathbb{Z})$ transformation can be found bringing the charges to a form where condition, eq. (3.34) is met, if two integers, c, d exist which are coprime, and which satisfy the condition, eq. (3.55). Generically, we have argued above, such integers do not exist, and thus eq. (3.34) will not be met.

One final comment before we move on. In the analysis above we considered the case where $\vec{Q}'^2_m$ had the same sign as $\vec{Q}^2_e$. If instead $\vec{Q}'^2_m$ has the opposite sign as $\vec{Q}^2_e$, the condition, eq. (3.32) is replaced by,

$$0 < d^2\alpha - \frac{c^2}{\alpha} \ll 1. \tag{3.58}$$

The discussion above, for the irrational and rational values of α, then goes through essentially unchanged leading to similar conclusions. For generic values of the charges, condition eq. (3.31) will not be satisfied. The condition in eq. (3.46) in this case is replaced by the requirement that

$$\alpha = \frac{m}{n} + \epsilon, \tag{3.59}$$

with $\epsilon > 0$, such that,

$$2n^2\epsilon \ll 1. \tag{3.60}$$

If this requirement is met, eq. (3.58) can be met by taking, $c = n, d = m$. For rational, α, eq. (3.55) is replaced by,

$$0 < \frac{2|c|}{m}(m|d| - n|c|) \ll 1. \tag{3.61}$$

3.4 Changing the charges

In our discussion above for the non-supersymmetric case we saw that for rather special values of α the condition, eq. (3.34) can be met. An example is given in eq. (3.45). This prompts one to ask the following question: Although a generic charge configuration cannot be brought to the Cardy limit, can we find a charge configuration lying near by, which can be brought to the Cardy limit ? In this subsection we will answer the question. For large charges, $Q \gg 1$, we show that such a near-by charge configuration does exist in the non-supersymmetric case. In contrast, in the supersymmetric case, such a near-by configuration does not exist.

Before proceeding let us state more clearly what we mean by a charge configuration lying near the starting $D0 - D4$ configuration. Suppose we carry out a change in the charges,

$$\vec{Q}_e \to \vec{Q}_e + \Delta\vec{Q}_e \tag{3.62}$$

$$\vec{Q}_m \to \vec{Q}_m + \Delta\vec{Q}_m. \tag{3.63}$$

The change is small, and the new charge configuration is near the original one, if the conditions,

$$\left|\frac{\vec{Q}_e \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| \ll 1$$
$$\left|\frac{\vec{Q}_m \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| \ll 1$$
$$\left|\frac{\Delta\vec{Q}_{e,m} \cdot \Delta\vec{Q}_{e,m}}{(\vec{Q}_{e,m})^2}\right| \ll 1, \tag{3.64}$$

are met.[10] In these inequalities, $\Delta\vec{Q}_{e,m}$ in the numerator stands for either, $\Delta\vec{Q}_e$, or $\Delta\vec{Q}_m$, the inequality holds in both cases. Similarly, $\vec{Q}_{e,m}$ in the denominator stands for either $\vec{Q}_e$ or $\vec{Q}_m$. Note that it follows from these conditions that the change in the duality invariant, I, eq. (2.7), and therefore also the change in the entropy, eq. (2.8), eq. (2.9), is small.

Let us first consider the supersymmetric case. The required condition for an $\mathrm{SL}(2,\mathbb{Z})$ transformation, eq. (3.5), to exist is that α, eq. (3.29), satisfies the condition, eq. (3.33). Suppose we start with generic charges, where $\alpha \leq 1$, but where condition eq. (3.33) is not met, and now carry out the change in the charges, eq. (3.62). The initial charges, $\vec{Q}_e, \vec{Q}_m$, are both either space-like or time-like, and since condition eq. (3.33) is not met, are roughly comparable in magnitude. It is then clear, and straightforward to verify explicitly, that small changes, meeting conditions, eq. (3.64), will not allow, eq. (3.4) to be met. We learn then that in the supersymmetric case there is no near by configuration - obtained by a small change in charges- which brings the charges to the Cardy limit.

Next we come to the non-supersymmetric case. Here one of the two vectors, $\vec{Q}_e, \vec{Q}_m$ is space-like and the other time-like, and this makes the analysis more involved, as we have already seen above. We will explicitly construct a new set of charges, close to the original one and show that it can be taken to the Cardy limit after a duality transformation. The construction will be based on the example, eq. (3.45), and will proceed in two steps. We will first find an altered set of charges for which an $\mathrm{SL}(2,\mathbb{Z})$ transformation meeting condition, eq. (3.34), exists. Then in the second step we will further alter these charges so that the $\mathrm{SL}(2,\mathbb{Z})$ transformation we have identified in the first step, followed by an appropriate $O(6,22,\mathbb{Z})$ transformation, brings this final set of altered charges to the Cardy limit. At both stages we will ensure that the changes in the charges are small and that the conditions, eq. (3.64), are met.

In the starting configuration, the $D0-D4$ brane charges are large, of order, Q, and roughly comparable, so that α satisfies condition, eq. (3.30), but $\alpha \not\ll 1$.

The first step: in the first step, we then change the $D0-D4$ charges (no new charges are excited at this stage) so that the new value of α is a rational, m/n. The change in α can be kept small,

$$\left|\alpha - \frac{m}{n}\right| < \epsilon, \tag{3.65}$$

[10] These conditions are manifestly invariant under the $O(6,22,\mathbb{Z})$ group. Once we choose a particular basis to write the initial charges as, $(\vec{Q}_e, \vec{Q}_m)$, there is no residual $\mathrm{SL}(2,\mathbb{Z})$ invariance left. The conditions, eq. (3.64), are written in this basis, and are in-effect also $\mathrm{SL}(2,\mathbb{Z})$ invariant.

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with,

$$\epsilon < 1, \tag{3.66}$$

if we take the integers, m, n to be sufficiently large,

$$m, n \sim O(1/\epsilon). \tag{3.67}$$

The required change in the charges is of order ΔQ where,

$$\frac{\Delta Q}{Q} \sim \frac{\Delta \alpha}{\alpha} \sim \epsilon \tag{3.68}$$

Next, we change one of the $D4$-brane charges by order unity, this gives rise to a final value of[11] α,

$$\alpha = \frac{m}{n}\sqrt{1-\frac{1}{Q}}. \tag{3.69}$$

Now choosing,

$$c = m, d = n, \tag{3.70}$$

eq. (3.37) is met, if the condition,

$$\frac{mn}{Q} < \delta, \tag{3.71}$$

is valid. Using eq. (3.67) this gives,

$$\epsilon > \frac{1}{\sqrt{\delta Q}}. \tag{3.72}$$

We will see below, that δ which was introduced first in eq. (3.37), can be taken to be a fixed small number, meeting condition, eq. (3.38), and not scaling like an inverse power of Q. Then by taking Q to be sufficiently big, so that

$$Q \gg \frac{1}{\delta} \gg 1, \tag{3.73}$$

condition eq. (3.72) can be made compatible with eq. (3.66). To keep the shift in the charges small, it is best to take ϵ to be as small as possible, subject to the condition, eq. (3.72). We will take,

$$\epsilon \sim \frac{1}{\sqrt{\delta Q}}. \tag{3.74}$$

It is useful in the subsequent discussion to distinguish between the altered charges obtained at this stage and the original charges we started with. We denote the altered charges by the tilde superscript. In the basis, eq. (2.4), eq. (2.5), we have,

$$\begin{aligned} \vec{\tilde{Q}}_e &= (\tilde{q_0}, -\tilde{p}^1, 0, 0, \cdots, 0) \\ \vec{\tilde{Q}}_m &= (0, 0, \tilde{p^i}, 0, 0, 0, 0). \end{aligned} \tag{3.75}$$

Before proceeding further it is worth examining condition eq. (3.71) more carefully. The inequality, eq. (3.34), arose from eq. (3.4). It's stronger form is given by the condition

[11] For example, if only , $p^2, p^3 \neq 0$, in the basis, eq. (2.5), then changing p^2 by unity would give, $\alpha = \frac{m}{n}\sqrt{1-\frac{1}{p^2}} = \frac{m}{n}\sqrt{1-\frac{1}{Q}}$, if $p^2 = Q$.

in eq. (3.3). Here, $p^{1'}$ is the charge that arised due to the $D4$-branes wrapping the K3, in the final configuration which lies in the Cardy limit and which is obtained by starting with the altered charges and doing the duality transformation. From eq. (3.3), eq. (3.37) we see that δ must satisfy the condition,

$$\delta \ll \frac{1}{|p^{1'}|}. \tag{3.76}$$

Now if $p^{1'} \sim Q$ we see that eq. (3.76), eq. (3.72), together imply that the condition in eq. (3.66) cannot be met. We will see below that the final charge configuration has a value for $p^{1'}$ which is much smaller than Q. In fact $p^{1'}$ can be taken to be $O(1)$ and not $O(Q)$. Thus, as was mentioned above, δ can be taken to be a small number not scaling like an inverse power of Q. One can then choose Q to meet the condition, eq. (3.73), and this will then suffice to meet eq. (3.72) and eq. (3.66).

From eq. (3.68) and eq. (3.74) we see that the required change in the charges are of the order,

$$\frac{\Delta Q}{Q} \sim \epsilon \sim \frac{1}{\sqrt{\delta Q}}. \tag{3.77}$$

This gives,

$$\Delta Q \sim \sqrt{\frac{Q}{\delta}}. \tag{3.78}$$

We see that while, $\Delta Q \gg 1$, from eq. (3.77), eq. (3.73), it follows that,

$$\frac{\Delta Q}{Q} \sim \frac{1}{\sqrt{\delta Q}} \ll 1, \tag{3.79}$$

so that the fractional change in the charges are small. Condition eq. (3.79) ensures that the requirements in eq. (3.64) are met, so that the changes in charge are small.

We have now completed the first step. The $\mathrm{SL}(2,\mathbb{Z})$ transformation that takes the altered charges to the Cardy limit has the form,

$$A = \begin{pmatrix} a & b \\ m & n \end{pmatrix} \tag{3.80}$$

The integers m, n have been determined in terms of α for the altered charges above eq. (3.69). As discussed in appendix B, a, b, can be chosen so that they satisfy the conditions,

$$\begin{aligned} a &\sim O(m) \\ b &\sim O(n). \end{aligned} \tag{3.81}$$

The relations in eq. (3.81) will be important in the following discussion.

The second step: we now proceed to the second step and construct the $O(6,22,\mathbb{Z})$ transformation. This will require a further change in the charges. We will excite extra charges which lie in the last two $\mathcal{H} \oplus \mathcal{H}$ subspaces in eq. (2.2). These are charges which arises from the T^2. The altered charges at the first stage are given in eq. (3.75). We now change them further, so that the final altered charges take the form,

$$\begin{aligned}\vec{\tilde{Q}}_e &= (\tilde{q}_0, -\tilde{p}^1, 0, 0, \cdots, -b, 0, n, 0)\\ \vec{\tilde{Q}}_m &= (0, 0, \tilde{p}^i, a, 0, -m, 0).\end{aligned} \tag{3.82}$$

Here a, b, m, n are elements of the $\mathrm{SL}(2, \mathbb{Z})$ matrix, eq. (3.80). Note that, $\tilde{q}_0, \tilde{p}^i \sim O(Q)$. From eq. (3.67), eq. (3.81), we see that $a, b, m, n \sim 1/\epsilon$. From, eq. (3.74) we then learn that

$$a, b, m, n \sim \frac{1}{\epsilon} \sim \sqrt{\delta Q}. \tag{3.83}$$

The changes in charges that give eq. (3.82) then meet the condition

$$\frac{\Delta Q}{Q} \sim \sqrt{\frac{\delta}{Q}} \ll 1, \tag{3.84}$$

where the last inequality follows from the fact that the charge Q meets the condition, eq. (3.73). This ensures that the conditions in eq. (3.64) are met.

The $\mathrm{SL}(2, \mathbb{Z})$ transformation, eq. (3.80), followed by an $O(6, 22, \mathbb{Z})$ transformation that we describe explicitly in appendix C, now brings the charges, eq. (3.82) to the form,

$$\begin{aligned}\vec{Q}'_e &= (a\tilde{q}_0, 1, b\tilde{p}^i, 0, -ma\tilde{q}_0\tilde{p}^1, 1, -a\tilde{q}_0(a\tilde{p}^1 + 1))\\ \vec{Q}'_m &= (m\tilde{q}_0, 0, n\tilde{p}^i, 1, -m^2\tilde{q}_0\tilde{p}^1, 0, -m(a\tilde{p}^1 + 1)\tilde{q}_0).\end{aligned} \tag{3.85}$$

These charges are in the Cardy limit. Since the second entry in $\vec{Q}'_m$ vanishes, the $D6$-brane charge vanishes. From the second entry in $\vec{Q}'_e$ we see that $|p^{1'}|$ is unity, as was promised above. Finally, the extra charges excited in going from eq. (3.75) to eq. (3.82) does not change the value of $(\vec{\tilde{Q}}_m)^2$. Thus,

$$\frac{(\vec{Q}'_m)^2}{\sqrt{|I|}} \simeq \left(\frac{mn}{Q}\right) \simeq \delta \ll 1, \tag{3.86}$$

where we have used eq. (3.67) for m, n and eq. (3.74) for ϵ. It then follows that eq. (3.3) is met and the final charges are in the Cardy limit.

Two comments before we end. First, there is some leeway in the $O(6, 22, Z)$ transformation which acting on the charges, eq. (3.82), brings them to the Cardy limit. For example, an $O(6, 22, \mathbb{Z})$ transformation can be found that results in $p^{1'}$ being a number much large than unity, but not scaling with Q. Second, we have seen in subsection 3.2 that in the vicinity of one set of charges which can brought to the Cardy limit, are other near by charges meeting condition, eq. (3.51), which can also be taken to the Cardy limit. Using, eq. (3.70), eq. (3.67), we see that eq. (3.51) takes the form,

$$\frac{\Delta Q}{Q} \sim \delta\epsilon^2. \tag{3.87}$$

Since, $\delta \ll 1, \epsilon < 1$, the size of this variation, $\frac{\Delta Q}{Q} \ll \epsilon$. Thus starting from one of the special charge configurations which can be brought to the Cardy limit, a variation of order, eq. (3.87), takes us to charges of the generic kind which can no longer be taken to the Cardy limit by a duality transformation. These charges will have to be changed by an amount of order, eq. (3.68), to be able to bring them to the Cardy limit.

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4. The $D0 - D6$ system

In this section we consider the $D0-D6$ system, where only $q_0, p^0 \neq 0$, and all other charges vanish, eq. (2.4), eq. (2.5). We show that such a charge configuration can never be brought to the Cardy limit. For this set of charges we have the following relations,

$$\begin{aligned} \vec{Q}_e^2 &= 0 \\ \vec{Q}_m^2 &= 0 \\ \vec{Q}_e \cdot \vec{Q}_m &= q_0 p^0. \end{aligned} \tag{4.1}$$

The invariant I, eq. (2.7), is,

$$I = -(q_0 p^0)^2, \tag{4.2}$$

It is negative, and the state breaks supersymmetry.

Let us assume that there is an $\mathrm{SL}(2,\mathbb{Z})$ transformation, eq. (3.5) which followed by an $O(6,22,\mathbb{Z})$ transformation brings the charges to the Cardy limit. Denoting the final charges by $\vec{Q}'_e, \vec{Q}'_m$, we have that,

$$\vec{Q}'^2_m = 2cdq_0 p^0. \tag{4.3}$$

If the final charges are in the Cardy limit, it follows from eq. (2.19), and the fact that $|p^{1'}| \geq 1$ that,

$$\frac{|(\vec{Q}'_m)^2|}{\sqrt{|I|}} \ll 1. \tag{4.4}$$

From, eq. (4.3) and eq. (4.2), this leads to the condition,

$$|cd| \ll 1. \tag{4.5}$$

Now note that c, d are integers. Thus the only way in which eq. (4.5) can be met is if $cd = 0$. This will mean that $\vec{Q}'^2_m = 0$ and hence the central charge, eq. (2.16), for the final charges vanishes. We do not want the central charge to vanish since the resulting AdS_3 space-time would not be described by weakly coupled supergravity. As a result we find that there is no duality transformation which can bring the $D0 - D6$ system to the Cardy limit.

In parallel with our discussion of section 3.4 we now ask if there are near by charges which can be brought to the Cardy limit. The following construction shows that such a set of charges does exits, as in the non-supersymmetric $D0 - D4$ system. The $D0 - D6$ system we start with has charges which in the basis, eq. (2.4), eq. (2.5), are given by,

$$\begin{aligned} \vec{Q}_e &= (q_0, 0, \cdots, 0) \\ \vec{Q}_m &= (0, p^0, 0, \cdots, 0). \end{aligned} \tag{4.6}$$

The charges meet the condition,

$$|\vec{Q}_e \cdot \vec{Q}_m| = |q_0 p^0| \gg 1. \tag{4.7}$$

For the change in the charges to be small the condition, analogous to eq. (3.64) in the $D0 - D4$ case, is given by,

$$\begin{aligned} \left|\frac{\vec{Q}_e \cdot \Delta\vec{Q}_{e,m}}{\vec{Q}_e \cdot \vec{Q}_m}\right| &\ll 1 \\ \left|\frac{\vec{Q}_m \cdot \Delta\vec{Q}_{e,m}}{\vec{Q}_e \cdot \vec{Q}_m}\right| &\ll 1 \\ \left|\frac{\Delta\vec{Q}_{e,m} \cdot \Delta\vec{Q}_{e,m}}{\vec{Q}_e \cdot \vec{Q}_m}\right| &\ll 1. \end{aligned} \tag{4.8}$$

Now consider the altered charges,

$$\begin{aligned} \vec{Q}_e &= (q_0, 0, 1, 0, \cdots, 0) \\ \vec{Q}_m &= (0, p^0, -1, 1, \cdots, 0). \end{aligned} \tag{4.9}$$

It is easy to see that conditions, eq. (4.8), are met and the changes in the charges are small.

In eq. (4.9), we have activated additional charges lying in the second Hyperbolic sublattice, $\mathcal{H}$, defined in eq. (2.2). We could have instead activated the additional charges to lie in any of the other Hyperbolic sublattices (or infact the $\mathcal{E}_8$ sublattices), and a similar discussion would go through.

Now consider an $O(2,2)$ transformation acting on the two $\mathcal{H}$ sublattices in which the charges lie, of the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & p^0 & 0 \\ 0 & 0 & 1 & 0 \\ -p^0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

This brings the altered charges, eq. (4.9), to the form,

$$\begin{aligned} \vec{Q}'_e &= (q_0, p^0, 1, -p^0 q_0, 0, \cdots, 0) \\ \vec{Q}'_m &= (0, 0, -1, 1, 0, \cdots 0). \end{aligned} \tag{4.11}$$

These charges are in the Cardy limit. The second entry in $\vec{Q}'_m$ vanishes, therefore, $p^{0'} = 0$. Also, $p^{1'} = p^0, (\vec{Q}'_m)^2 = -2$, so that the condition, eq. (2.19), is met, as long as

$$|q_0| \gg 1. \tag{4.12}$$

Note that the central charge, $C \sim |p^{1'}(\vec{Q}'_m)^2| \sim (p^0)^2$. This meets the condition, $C \gg 1$ if $|p^0| \gg 1$. Alternatively, if $p^0 \sim O(1)$, we can excite additional charges in eq. (4.9) so that, for example, $p^{1'} \gg 1$, and thus $C \gg 1$.

5. Absence of magnetic monopole charge

We have mentioned above that lifting a configuration with $D6$ brane charge to M-theory cannot give a locally AdS_3 spacetime in the near-horizon limit. We prove this statement here.

We start with a general extremal black hole, carrying charges given in eq. (2.4), eq. (2.5), in four dimensions in IIA theory. The near horizon geometry is $AdS_2 \times S^2$. An AdS_2 space-time has SO(2,1) symmetry. This gets enhanced to SO(2,2) in the AdS_3 case.[12] In the special case where the black hole carries no $D0$-brane charge, N units of $D6$-brane charge, and arbitray values of the other charges, it is well known that one does not get the SO(2,2) symmetry of AdS_3 in the near horizon limit geometry. The $D6$-brane charge is KK monopole charge along the M direction. This charge results in the M-direction being fibered over the S^2 resulting in the near horizon geometry of form, $AdS_2 \times S^3/Z_N$.

Here we will examine what happens if the black hole carries both $D0$ and $D6$ brane charges, besides having arbitrary values of the other charges, and find that the symmetries of the near horizon geometry are $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{U}(1)$ and are therefore not enhanced to SO(2,2). This proves that the only way to get a locally AdS_3 geometry on lifting to M-theory is for the $D6$-brane charge to vanish.

Lifting the $AdS_2 \times S^2$ near-horizon geometry to M-theory, gives,

$$\begin{aligned} ds^2 &= R^2(-\cosh^2\theta_1 d\phi_1^2 + d\theta_1^2) + R^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &\quad + g_{\psi\psi}(d\psi + \alpha \sinh\theta_1 d\phi_1 + \beta\cos\theta_2 d\phi_2)^2 \end{aligned} \tag{5.1}$$

Here we are using Global coordinates θ_1, ϕ_1 for AdS_2, polar coordinates, θ_2, ϕ_2 for the S^2, and denoting the M-theory direction as ψ. The metric component, $g_{\psi\psi}$, is a constant. α, β are proportional to the $D0$ and $D6$ brane charges and are non-vanishing if these charges are non-vanishing. We seek the Killing vectors for this metric.

It is convenient to analytically continue the AdS_2 metric to that of S^2 as follows,

$$\begin{aligned} \theta_1 &\to i\left(\frac{\pi}{2} - \theta_1\right) \\ (R^2)_{\mathrm{AdS}} &\to -R^2 \\ \alpha &\to -i\alpha. \end{aligned} \tag{5.2}$$

This gives,

$$\begin{aligned} ds^2 &= R^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + R^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &\quad + g_{\psi\psi}(d\psi + \alpha\cos\theta_1 d\phi_1 + \beta\cos\theta_2 d\phi_2)^2. \end{aligned} \tag{5.3}$$

We show that the isometry group of this metric is, $\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{U}(1)$, it will then follow by analytic continuation that the isometry group of eq. (5.1) is, $\mathrm{SO}(2,1) \times \mathrm{SO}(3) \times \mathrm{U}(1)$.

By rescaling the ψ coordinate, α and β, this metric can be written as,

$$\begin{aligned} ds^2 &= R^2[(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + (d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &\quad + (d\psi' + \alpha'\cos\theta_1 d\phi_1 + \beta'\cos\theta_2 d\phi_2)^2]. \end{aligned} \tag{5.4}$$

α', β' are proportional to α, β and only vanish when the latter do. Next we drop the overall factor of R^2, and rescale ϕ_1, ϕ_2 as follows,

$$\alpha'\phi_1 \to \phi_1, \quad \beta'\phi_2 \to \phi_2. \tag{5.5}$$

[12] Our analysis of the symmetries in this section will be local. So the breaking of SO(2,2) symmetry due to identifications which are made in the BTZ geometry will not be relevant.

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Note this rescaling is well defined only if α', β', and hence α, β, are non-vanishing. This gives for the metric,

$$\begin{aligned} ds^2 =& \, d\theta_1^2 + d\theta_2^2 + (1+(\tilde{\alpha})^2 \sin^2\theta_1) d\phi_1^2 + (1+(\tilde{\beta})^2 \sin^2\theta_2) d\phi_2^2 + d\psi^2 \\ & + 2\cos\theta_1 d\psi d\phi_1 + 2\cos\theta_2 d\psi d\phi_2 + 2\cos\theta_1 \cos\theta_2 d\phi_1 d\phi_2, \end{aligned} \tag{5.6}$$

where,

$$(\tilde{\alpha})^2 = \frac{1}{\alpha'^2} - 1 \tag{5.7}$$

$$(\tilde{\beta})^2 = \frac{1}{\beta'^2} - 1. \tag{5.8}$$

To save clutter we will henceforth drop the tildes on α, β and denote the metric in eq. (5.6) as,

$$\begin{aligned} ds^2 =& \, d\theta_1^2 + d\theta_2^2 + (1+\alpha^2 \sin^2\theta_1) d\phi_1^2 + (1+\beta^2 \sin^2\theta_2) d\phi_2^2 + d\psi^2 \\ & + 2\cos\theta_1 d\psi d\phi_1 + 2\cos\theta_2 d\psi d\phi_2 + 2\cos\theta_1 \cos\theta_2 d\phi_1 d\phi_2. \end{aligned} \tag{5.9}$$

The reader should note that α, β, in eq. (5.9) are different from α, β, as appearing in eq. (5.3).

We now turn to studying the isometries of the metric, eq. (5.9). First note that $\partial_{\phi_1}, \partial_{\phi_2}, \partial_\psi$, are commuting isometries of this metric. They can be taken to be part of the Cartan generators of the full isometry group. Any other killing vector, ξ, can then be taken to carry definite charges with respect to these generators, and satisfies the relations,

$$[\partial_{\phi_1}, \xi] = i m_1 \xi, \tag{5.10}$$

$$[\partial_{\phi_2}, \xi] = i m_2 \xi, \tag{5.11}$$

$$[\partial_\psi, \xi] = i m_3 \xi, \tag{5.12}$$

where m_1, m_2, m_3 are the eigenvalues with respect to these three isometries.

The killing vector, ξ, must satisfy the Killing conditions,

$$\partial_\alpha \xi^\gamma g_{\gamma\beta} + \partial_\beta \xi^\gamma g_{\gamma\alpha} + \xi^\gamma \partial_\gamma g_{\alpha\beta} = 0 \tag{5.13}$$

for all values of α, β.

These Killing conditions are studied in more detail in appendix D. One finds that there are only four more non-trivial Killing vectors, corresponding to $m_1 = \pm\sqrt{1+\alpha^2}, m_2 = m_3 = 0$ and $m_2 = \pm\sqrt{1+\beta^2}, m_1, m_3 = 0$. Altogether there are then seven Killing vectors, given by,

$$\begin{aligned} \xi_1 &= e^{i\sqrt{1+\alpha^2}\phi_1} \left[\partial_{\theta_1} + \frac{i}{\sqrt{1+\alpha^2}} \cot\theta_1 \partial_{\phi_1} - \frac{i}{\sqrt{1+\alpha^2}} \frac{1}{\sin\theta_1} \partial_\psi \right] \\ \xi_2 &= e^{-i\sqrt{1+\alpha^2}\phi_1} \left[\partial_{\theta_1} - \frac{i}{\sqrt{1+\alpha^2}} \cot\theta_1 \partial_{\phi_1} + \frac{i}{\sqrt{1+\alpha^2}} \frac{1}{\sin\theta_1} \partial_\psi \right] \\ \xi_3 &= \partial_{\phi_1} \end{aligned}$$

$$\begin{aligned}
\xi_4 &= e^{i\sqrt{1+\beta^2}\phi_2}\left[\partial_{\theta_2} + \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} - \frac{i}{\sqrt{1+\beta^2}}\frac{1}{\sin\theta_2}\partial_\psi\right] \\
\xi_5 &= e^{-i\sqrt{1+\beta^2}\phi_2}\left[\partial_{\theta_2} - \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} + \frac{i}{\sqrt{1+\beta^2}}\frac{1}{\sin\theta_2}\partial_\psi\right] \\
\xi_6 &= \partial_{\phi_2} \\
\xi_7 &= \partial_\psi
\end{aligned} \tag{5.14}$$

The first three give rise to an SO(3) isometry, the second three to another SO(3) and the last to an U(1) isometry, giving the total symmetry group, SO(3) $\times$ SO(3) $\times$ U(1). After analytic continuation this implies that the metric we started with has isometries, SO(2, 1) $\times$ SO(3) $\times$ U(1).

We refer the reader to appendix D for more details.

6. Conclusions

This paper has two main results. First, we have shown that a generic supersymmetric or non-supersymmetric system of charges cannot be brought to the Cardy limit using the duality symmetries. Second, we have found that the required non-genericity to be able to bring a set of charges to the Cardy limit is interestingly different in the supersymmetric and the non-supersymmetric cases. For large charge, in the non-supersymmetric case but not the supersymmetric one, we can always find a set of charges lying close by which can be brought to the Cardy limit. The required shift in the charges satisfy the condition,[13]

$$\frac{\Delta Q}{Q} \sim \frac{1}{\sqrt{Q}}. \tag{6.1}$$

These results were proved for the $D0-D4$ system and the $D0-D6$ system. We expect them to be more general.

For example, our analysis of the $D0-D4$ system, leading to the conclusion that generic charges cannot be brought to the Cardy limit, immediately applies to all charges which satisfy the condition,

$$\vec{Q}_e \cdot \vec{Q}_m = 0. \tag{6.2}$$

Similarly, the analysis of the $D0-D6$ system applies to all charges meeting the condition,

$$\vec{Q}_e^2 = \vec{Q}_m^2 = 0. \tag{6.3}$$

with the conclusion that all such charges can never be brought to the Cardy limit. Also, all the results immediately apply to other charges which lie in the same duality orbit as the D0-D4 or D0-D6 systems.

In our analysis we did not determine all the necessary and sufficient conditions that need to be met to be able to bring a set of charges to the Cardy limit. To obtain a more complete understanding of these conditions, for a general set of charges, it would be useful

[13] More correctly, the condition in the $D0-D4$ case is given in eq. (3.77), where δ is a small number that does not scale with Q, and in the $D0-D6$ case, with $q_0, p^0 \gg 1$, it is given by, $\frac{\Delta Q}{Q} \sim \frac{1}{Q}$.

to start with a classification of all the discrete invariants of $\mathrm{SL}(2,\mathbb{Z})\times O(6,22,\mathbb{Z})$. It should be possible to express the required conditions, for any charge configuration to be brought to the Cardy limit, in terms of these invariants. We leave such an analysis for the future.

Another approach would be to bring the charges to a canonical form and then carry out the analysis for general charges of this form. As long as the charges lie in the $\Gamma^{(6,6)}$ sublattice, made out of the 6 Hyperbolic sublattices, $\mathcal{H}$ in eq. (2.2), one can show using the duality symmetries that the electric charges, $\vec{Q}_e$, can always be made to lie only in first hyperbolic sublattice, while the magnetic charges, $\vec{Q}_m$, take non-trivial values in the first two hyperbolic sublattices. These results are discussed in appendix E. One expects these results to be further generalised, when charges lying in the $\mathcal{E}_8\times\mathcal{E}_8$ sublattice are also excited. For example, it has shown that a general time-like vector can always be made to lie in one Hyperbolic sublattice, (see the discussion in[14] [36]). Further analysis along these lines is also left for the future.

Our conclusions in the supersymmetric case are in accord with recent results obtained for the subleading corrections to the entropy, going like $1/Q$. If the system could be brought to the Cardy limit these corrections would be of the form, eq. (2.14), with the central charge receiving $1/Q$ corrections. The results for the first subleading corrections, which have been obtained by directly counting the dyonic degeneracy and computing the four derivative corrections using the Gauss-Bonnet term, are now known not to be generally of this form, [11, 37, 38].

One of the main motivations of this investigation was to ask how far the AdS_3/CFT description can take us in understanding the entropy of non-supersymmetric black holes. If the charges lie in the Cardy limit, then at least in some region of moduli space, the black hole with these charges can be viewed as a BTZ black hole in AdS_3 space. The microscopic states which account for the black hole entropy can then be understood as states in a $1+1$ dim. CFT, and their entropy can be easily found in terms of the Cardy formula. Our result, that in the non-supersymmetric case a generic set of charges, after a small shift, can be brought to the Cardy limit is quite promising in this context. It tells us that such a microscopic counting for the leading order entropy is available for generic charges, at least in some region of moduli space.

The main complication in determining the entropy microscopically is then it's possible moduli dependence. This is a particularly important issue in the non-supersymmetric case. In the Cardy formula the entropy is determined by the central charge. Now, the central charge is protected by anomaly considerations and is therefore moduli independent. Thus for the charges which can be brought to the Cardy limit, the entropy must be moduli independent, at least for small shifts of moduli.[15] Since the required fractional shift to get to such a configuration is small, of order, $O(1/\sqrt{Q})$, eq. (6.1), one would hope that this is enough to prove that the leading entropy is generally moduli independent.

Once the moduli independence of the entropy is established, it is easy to furnish an argument, as follows, leading to the determination of the entropy microscopically. The

[14] Also, V.V.Nikulin, Math.USSR Izvestija,14(1980),pg.103.

[15] Larger shifts might result in a jump, akin to a phase transition, where the formula for the entropy gets significant corrections.

entropy must now be a function only of the charges. And the dependence on the charges must enter through invariants of the discrete duality group, which is an exact symmetry of string theory. For the case we are studying here, one of these invariants, I, eq. (2.7), is also an invariant of the full continuous group, $\mathrm{SL}(2,\mathbb{R}) \times O(6,22,\mathbb{R})$. The others are discrete invariants. Now the discrete invariants are not continuous functions of charge and typically undergo big jumps when the charges are changed only slightly[16] It is physically reasonable to demand that for large charges the leading order entropy does not undergo such discontinuous jumps. This would mean that any dependence on the discrete invariants must be subdominant at large charge.[17] The resulting functional dependence on the continuous invariant can then be determined by taking any convenient set of charges, which gives rise to a non-vanishing value for this invariant. In particular one can always find charges in the Cardy limit for which this invariant does not vanish. For such a set of charges a microscopic calculation of the entropy is often possible as was mentioned above, and this would then determine the entropy for all general charges.

These arguments should also apply when one includes angular momentum in four dimensions, $\vec{J}$. In this case there are now two invariants of the continuous duality symmetries, and the Rotation group, I and $\vec{J}^2$. An argument along the above lines would fix the dependence on both these invariants. Note that the resulting expression for the entropy would then also be valid when I, and more generally all the charges, $\vec{Q}_e, \vec{Q}_m$ vanish, leading to microscopic determination of the entropy of an extreme Kerr black hole in four dimensions. It is easy to check that the resulting answer is in agreement with the Beckenstein-Hawking entropy in this case.

These arguments will be developed, at more length and with more care, in a forthcoming paper.

The arguments above, whose purpose is to provide a microscopic understanding of the entropy, are already known to have counterparts on the gravity side. This makes us hopeful that they can be more fully fleshed out on the microscopic side as well. We end with a brief discussion of these issues from the gravity point of view.

Recent advances have now established that the attractor mechanism is valid for all extremal black holes, supersymmetric as well as non-supersymmetric ones (See [39 − 42], for early work. More recent advances are in, e.g, [43, 44, 37, 45 − 63], see also, [64], and references therein). This shows that the entropy is not dependent on the moduli.[18] Once the moduli independence is established the duality symmetries allow the entropy for general charges to be related to the entropy which arise for a set of charges in the Cardy limit. In the supergravity approximation, which is valid at large charge, the duality group is enhanced

[16]For example consider the discrete invariant, $\gcd(Q_e^i Q_m^j - Q_e^j Q_m^i, Q_e^k Q_m^l - Q_e^l Q_m^k), \forall i,j,k,l \in \{1,2,\cdots,28\}$. Since the gcd can vary discontinuously, this invariant can change by big jumps.

[17]This argument was given to us by Shiraz Minwalla, we thank him for the discussion on this point and related issues.

[18]More correctly this shows that the entropy is independent of small shifts in the moduli. There can be discontinuous jumps in the entropy as the moduli are varied, see ref Moore and Denef for related recent developments. However, this might be less of a worry if we are interested in the entropy of a single-centered black hole.

to the full continuous group, in the case we are considering here to $\mathrm{SL}(2,\mathbb{R})\times O(6,22,\mathbb{R})$. A duality transformation will act on both the charges and the moduli, and to begin with the entropy could have been a duality invariant function of the moduli and charges. However, once we have established that the entropy is moduli independent it must be an invariant of the charges alone. Since there is only one duality invariant of the continuous group,[19] I, the entropy for a general set of charges can be related to the entropy for charges in the Cardy limit, with the same value of this invariant.

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A. Tightening the conditions in the supersymmetric case

A supersymmetric $D0-D4$ system, which can be taken to the Cardy limit, must meet the condition, eq. (3.15). In this appendix we show that this condition can be somewhat strengthened, leading to eq. (3.16).

This comes about as follows. In general the $\mathrm{SL}(2,\mathbb{Z})$ transformation, eq. (3.13), will be followed by an $O(6,22,\mathbb{Z})$ transformation, $B\in O(6,22,\mathbb{Z})$, to obtain the final configuration, $(\vec{Q}'_e,\vec{Q}'_m)$ which is given by,

$$\vec{Q}'_e = B\vec{Q}_e + bB\vec{Q}_m \tag{A.1}$$

$$\vec{Q}'_m = B\vec{Q}_m. \tag{A.2}$$

We will see shortly that this final configuration is in the Cardy limit if and only if the configuration, $(\tilde{\vec{Q}}_e,\tilde{\vec{Q}}_m)$, defined by,

$$(\tilde{\vec{Q}}_e,\tilde{\vec{Q}}_m) = (B\vec{Q}_e, B\vec{Q}_m) \tag{A.3}$$

is in the Cardy limit. Note that the charges, $(\tilde{\vec{Q}}_e,\tilde{\vec{Q}}_m)$, are obtained by applying only the transformation, $B\in O(6,22,\mathbb{Z})$ on $(\vec{Q}_e,\vec{Q}_m)$. Applying condition eq. (3.2) to the charges, $(\tilde{\vec{Q}}_e,\tilde{\vec{Q}}_m)$, we learn that for them to be in the Cardy limit,

$$|I| \gg \left(\tilde{p}^1\left(\tilde{\vec{Q}}_m\right)^2\right)^2. \tag{A.4}$$

[19] We are neglecting angular momentum, $\vec{J}$, here.

From eq. (A.3) we see that $\left(\vec{\tilde{Q}}_m\right)^2 = \vec{Q}_m^2$. Now since $\vec{\tilde{Q}}_e$ is obtained by applying an $O(6,22,\mathbb{Z})$ transformation to $\vec{Q}_e$, the minimum value $\tilde{p}^1$ can take is $\gcd(\vec{Q}_e)$. Eq. (3.16) then follows, after using eq. (3.9) for I.

To complete the argument let us show that $(\vec{Q}'_e, \vec{Q}'_m)$ can be in the Cardy limit if an only if $(\vec{\tilde{Q}}_e, \vec{\tilde{Q}}_m)$ is in the Cardy limit. To see this we note that from eq. (A.1) and eq. (A.3) it follows that,

$$\vec{Q}'_e = \vec{\tilde{Q}}_e + b\vec{\tilde{Q}}_m, \tag{A.5}$$

and,

$$\vec{Q}'_m = \vec{\tilde{Q}}_m. \tag{A.6}$$

If $\vec{Q}'_m$ is in the Cardy limit the $D6$-brane charge for this configuration must vanish, so, $p^{0'} = 0$. From eq. (A.6) we see this implies that $\tilde{p}^0$ also vanishes. Eq. (A.6) also implies that $(\vec{Q}'_m)^2 = (\vec{\tilde{Q}}_m)^2$. And eq. (A.5) implies that $p^{1'} = \tilde{p}^1$. The second condition for the Cardy limit, eq. (2.19), is

$$I \gg 6(p'_1 \vec{Q}'^2_m)^2. \tag{A.7}$$

Since I is a duality invariant, it then follows that the condition eq. (A.7) is the same as the corresponding condition in terms of the tilde variables,

$$I \gg 6\left(\tilde{p_1}\left(\vec{\tilde{Q}}_m\right)^2\right)^2. \tag{A.8}$$

B. Some non-supersymmetric examples

In this appendix we present some examples of charges in th non-supersymmetric case, which can be brought to the Cardy limit after a duality transformation.

We take,

$$\vec{Q}_e = (p-1, -1, 0, 0, 0, \cdots 0) \tag{B.1}$$

$$\vec{Q}_m = (0, 0, 1, p, 0, \cdots 0), \tag{B.2}$$

with,

$$p \gg 1. \tag{B.3}$$

The quartic invariant, I, eq. (2.7) is,

$$I = -4p(p-1). \tag{B.4}$$

The value of $p^1 = 1$, and $\vec{Q}_m^2 = 2p$, so we see that condition, eq. (2.19) is not met and the starting configuration is not in the Cardy limit. In this example, $|\vec{Q}_e^2| < |\vec{Q}_m^2|$, so that $\alpha > 1$ to begin, we therefore carry out the SL$(2,\mathbb{Z})$ transformation, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which gives,

$$\vec{Q}_e = (0, 0, 1, p, 0, \cdots) \tag{B.5}$$

$$\vec{Q}_m = -(p-1, -1, 0, 0, 0, \cdots, 0). \tag{B.6}$$

The resulting value of α is,

$$\alpha = \sqrt{\frac{p-1}{p}}. \tag{B.7}$$

This is of the form discussed above in eq. (3.44). Starting with the charges, eq. (B.5), we now carry out $\mathrm{SL}(2,\mathbb{Z}) \times O(6,22,\mathbb{Z})$ transformations which bring it in the Cardy limit. The $\mathrm{SL}(2,\mathbb{Z})$ transformation is,

$$A = \begin{pmatrix} (p-1) & -p \\ 1 & -1 \end{pmatrix} \tag{B.8}$$

with resulting charges,

$$\vec{\tilde{Q}}_e = (p(p-1), -p, p-1, (p-1)p, 0\cdots, 0) \tag{B.9}$$
$$\vec{\tilde{Q}}_m = (p-1, -1, 1, p, 0, \cdots, 0) \tag{B.10}$$

This is followed by an $O(6,22,\mathbb{Z})$ transformation,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{B.11}$$

By this we mean that B acts non-trivially on the 4 dimensional sublattice of charges where the inner product is given by the first two factors of $\mathcal{H}$ in eq. (2.1), and acts trivially on the rest of the lattice. The transformation B gives the final charges,

$$\vec{Q}'_e = (p(p-1), -1, p-1, 0, 0, \cdots, 0) \tag{B.12}$$
$$\vec{Q}'_m = (p-1, 0, 1, 1, 0, \cdots 0). \tag{B.13}$$

Since the second entry in $\vec{Q}'_m$ vanishes, the $D6$ brane charge in the final configuration vanishes as is needed for the Cardy limit. From the second entry in $\vec{Q}'_e$ we see that $|p^{1'}| = 1$, and we also have that, $|\vec{Q}'^2_m| = 2$. Since I is given by, eq. (B.4), we see that condition eq. (2.19) is now met and the final set of charges are in the Cardy limit.

To obtain an example with all final charges which are non-zero being much bigger than unity we can scale the initial charges, so that $(\vec{Q}_e, \vec{Q}_m) \to (\lambda\vec{Q}_e, \lambda\vec{Q}_m), \lambda \gg 1$, and now take,

$$p \gg \lambda. \tag{B.14}$$

Another example is as follows. We take,

$$\vec{Q}_e = (q_0, -p^1, 0, 0, \cdots, 0) \tag{B.15}$$
$$\vec{Q}_m = (0, 0, p^2, p^2, 0, \cdots, 0), \tag{B.16}$$

with

$$|q_0| \sim |p^1|. \tag{B.17}$$

This system is not in the Cardy limit.

Applying the $O(6, 22)$ transformation which acts non-trivially only on the 4 dimensional sublattice gives by the first two factors of $\mathcal{H}$ in eq. (2.1) and has the form,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{B.18}$$

gives the final charges,

$$\vec{Q}'_e = (q_0, q_0 - p^1, q_0, -q_0, 0 \cdots 0) \tag{B.19}$$

$$\vec{Q}'_m = (0, 0, p^2, p^2, 0, \cdots). \tag{B.20}$$

As long as the condition,

$$|q_0 p^1| \gg 6(p_1 - q_0)^2 (p^2)^2 \tag{B.21}$$

is met this final configuration satisfies eq. (2.19) and is in the Cardy limit.

C. More details on changing the charges

Two results of relevance to section 3.4 will be derived here.

First, we show that an SL$(2, \mathbb{Z})$ matrix of the form, eq. (3.80), can always be found where a, b meet the conditions, eq. (3.81).

The integers, m, n are determined in terms of the value of α for the altered charges, eq. (3.69). These can be taken to be coprime. Thus an SL$(2, \mathbb{Z})$ matrix can always be found of the form,

$$A' = \begin{pmatrix} a' & b' \\ m & n \end{pmatrix} \tag{C.1}$$

The integers, a', b' satisfy the condition,

$$\det(A) = a'n - b'm = 1. \tag{C.2}$$

From here it follows that,

$$\left[\frac{a'}{m}\right] = \left[\frac{b'}{n}\right] \tag{C.3}$$

where $[\frac{a'}{m}]$ denotes the integer part of $|\frac{a'}{m}|$, and similarly for $[\frac{b'}{n}]$. Now, the allowed values of integers, a', b', which satisfy eq. (C.2) are not unique. One can see that if a', b' satisfy eq. (C.2) then so do,

$$a = a' - \left[\frac{a'}{m}\right] m \tag{C.4}$$

$$b = b' - \left[\frac{a'}{m}\right] n \tag{C.5}$$

From eq. (C.3) it follows that the relations in eq. (3.81) are valid. The resulting SL$(2, \mathbb{Z})$ transformation is then given in eq. (3.80).

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Next we show that starting with the charges, eq. (3.82), and applying the $\mathrm{SL}(2,\mathbb{Z})$ transformation, eq. (3.80), followed by an $O(6,22,\mathbb{Z})$ transformation, gives rise to the charges, eq. (3.85). The $\mathrm{SL}(2,\mathbb{Z})$ transformation acting on eq. (3.82) gives the charges,

$$\begin{aligned} \vec{\hat{Q}}_e &= (a\tilde{q}_0, -a\tilde{p}^1, b\tilde{p}^i, 0, 0, 1, 0) \\ \vec{\hat{Q}}_m &= (m\tilde{q}_0, -m\tilde{p}^1, n\tilde{p}^i, 1, 0, 0, 0). \end{aligned} \tag{C.6}$$

Next, we determine the $O(6,22,\mathbb{Z})$ transformation. Consider a four dimensional subspace of the charge lattice, where the metric, eq. (2.2), is, $\mathcal{H}\oplus\mathcal{H}$. The following matrix is an element of $O(2,2,\mathbb{Z})$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & 0 \\ -q & 0 & 0 & 1 \end{pmatrix}, \tag{C.7}$$

for any $q \in \mathbb{Z}$. Now starting with the charges, eq. (C.6), consider such a transformation, with $q = m\tilde{p}^1$, acting on the charges lying in the first Hyperbolic subspace and the second last Hyperbolic subspace, as defined in eq. (2.2). And next such a transformation, with $q = (a\tilde{p}^1+1)$, acting on the charges in the first Hyperbolic subspace and the last Hyperbolic subspace, as defined in eq. (2.2). This takes the charges in eq. (C.6) to their final values in eq. (3.85).

D. Some more details on the isometry analysis of section 5

In this section we will derive all the isometries preserved by the metric eq. (5.9). The Killing vectors must satisfy the conditions given by eq. (5.13). The $(\theta_1,\theta_1), (\theta_2,\theta_2), (\theta_1,\theta_2)$ components of this equation take the form,

$$\begin{aligned} \partial_{\theta_1}\xi^{\theta_1} &= 0 \\ \partial_{\theta_2}\xi^{\theta_2} &= 0 \\ \partial_{\theta_1}\xi^{\theta_2} + \partial_{\theta_2}\xi^{\theta_1} &= 0. \end{aligned} \tag{D.1}$$

The $(\phi_1,\phi_1), (\phi_2,\phi_2), (\phi_1,\phi_2)$, components are,

$$\begin{aligned} im_1\xi_{\phi_1} + \alpha^2\xi^{\theta_1}\sin\theta_1\cos\theta_1 &= 0 \\ im_2\xi_{\phi_2} + \beta^2\xi^{\theta_2}\sin\theta_2\cos\theta_2 &= 0 \\ im_1\xi_{\phi_2} + im_2\xi_{\phi_1} - \sin\theta_1\cos\theta_2\xi^{\theta_1} - \sin\theta_2\cos\theta_1\xi^{\theta_2} &= 0 \end{aligned} \tag{D.2}$$

The $(\psi,\psi), (\psi,\phi_1), (\psi,\phi_2)$, components are,

$$\begin{aligned} im_3\xi_\psi &= 0 \\ im_1\xi_\psi + im_3\xi_{\phi_1} - \sin\theta_1\xi^{\theta_1} &= 0 \\ im_2\xi_\psi + im_3\xi_{\phi_2} - \sin\theta_2\xi^{\theta_2} &= 0 \end{aligned} \tag{D.3}$$

The $(\theta_1,\phi_1), (\theta_2,\phi_2), (\theta_1,\phi_2), (\theta_2\phi_1)$, components are,

$$\partial_{\theta_1}\xi^\gamma g_{\gamma\phi_1} + im_1\xi^{\theta_1} = 0$$

$$\begin{aligned}\partial_{\theta_2}\xi^\gamma g_{\gamma\phi_2} + im_2\xi^{\theta_2} &= 0\\ \partial_{\theta_1}\xi^\gamma g_{\gamma\phi_2} + im_2\xi^{\theta_1} &= 0\\ \partial_{\theta_2}\xi^\gamma g_{\gamma\phi_1} + im_1\xi^{\theta_2} &= 0\end{aligned} \tag{D.4}$$

Finally the $(\theta_1, \psi), (\theta_2, \psi)$, components are,

$$\begin{aligned}\partial_{\theta_1}\xi^\gamma g_{\gamma\psi} + im_3\xi_1^\theta &= 0\\ \partial_{\theta_2}\xi^\gamma g_{\gamma\psi} + im_3\xi_2^\theta &= 0\end{aligned} \tag{D.5}$$

Setting $m_1 = m_2 = m_3 = 0$ we have from the (ψ, ϕ_1) and (ψ, ϕ_2) components that, $\xi_1^\theta = \xi_2^\theta = 0$. It then follows from the remaining equations that there are only three Killing vectors of this type. These are, $\partial_\psi, \partial_{\phi_1}, \partial_{\phi_2}$, which have already been identified above.

Next setting $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ we have, from the equation for (ψ, ψ), (ϕ_1, ϕ_1) and (ψ, ϕ_1) components that,

$$-\frac{\alpha^2\cos\theta_1}{m_1}\xi_1^\theta = \frac{1}{m_3}\xi_1^\theta, \tag{D.6}$$

from which we conclude that

$$\xi_1^\theta = 0. \tag{D.7}$$

Similarly we learn that $\xi_2^\theta = 0$. From the $(\phi^1, \phi^1), (\phi^2, \phi^2), (\psi, \psi)$, components it then follows that,

$$\xi_\mu = 0 \;\; \forall\, \mu, \tag{D.8}$$

leading to the conclusion that there is no Killing vector of this type.

We will now set $m_1 = m_2 = 0$ and $m_3 \neq 0$. The (ϕ_1, ϕ_1) and (ϕ_2, ϕ_2) components give, respectively, $\xi^{\theta_1} = 0$ and $\xi^{\theta_2} = 0$. The (ψ, γ) components for $\gamma = \psi, \phi_1$ and ϕ_2 give $\xi_\psi = 0, \xi_{\phi_1} = 0$ and $\xi_{\phi_2} = 0$ respectively. Thus we have no killing vector with $m_1 = m_2 = 0$ and $m_3 \neq 0$.

Let us now set $m_2 = m_3 = 0$ and $m_1 \neq 0$. Considering the (ϕ_2, ϕ_2) component, we get $\xi^{\theta_2} = 0$. From the $(\phi_1, \phi_1), (\phi_1, \phi_2)$ and (ψ, ϕ_1) components we get,

$$\begin{aligned}\xi_{\phi_1} &= -\left(\frac{\xi^{\theta_1}}{im_1}\right)\alpha^2\sin\theta_1\cos\theta_1\\ \xi_{\phi_2} &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\sin\theta_1\cos\theta_2\\ \xi_\psi &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\sin\theta_1\ .\end{aligned} \tag{D.9}$$

The contravariant components of ξ can be shown to be

$$\begin{aligned}\xi^{\phi_1} &= -\left(\frac{\xi^{\theta_1}}{im_1}\right)\cot\theta_1\\ \xi^\psi &= \left(\frac{\xi^{\theta_1}}{im_1}\right)\mathrm{cosec}\ \theta_1\,,\end{aligned} \tag{D.10}$$

and $\xi^{\phi_2} = 0$. We still have to satisfy the remaining nontrivial equations. The (θ_1, ϕ_1) component of the killing equation

$$\partial_{\theta_1}\xi^{\phi_1} g_{\phi_1\phi_1} + \partial_{\theta_1}\xi^{\psi} g_{\psi\phi_1} + im_1\xi^{\theta_1} = 0\,, \tag{D.11}$$

gives

$$-\frac{1}{m_1}(1+\alpha^2) + m_1 = 0\ . \tag{D.12}$$

Thus we must have

$$m_1 = \pm\sqrt{1+\alpha^2}\ . \tag{D.13}$$

It is straightforward to check that the (θ_1, ϕ_2) and (θ_1, ψ) components of the killing equation are satisfied. All other components are satisfied trivially provided ξ^{θ_1} is independent of θ_1, θ_2. As a result we get two linearly independent killing vectors corresponding to the two roots of m_1:

$$\begin{aligned}
\xi_1 &= e^{i\sqrt{1+\alpha^2}\phi^1}\left(\partial_{\theta_1} + \frac{i}{\sqrt{1+\alpha^2}}\cot\theta_1\partial_{\phi_1} - \frac{i}{\sqrt{1+\alpha^2}}\text{cosec}\ \theta_1\partial_\psi\right)\,,\\
\xi_2 &= \xi_1^*\ .
\end{aligned} \tag{D.14}$$

In a similar way we can obtain two more linearly independent killing vectors upon setting $m_1 = m_3 = 0$ and $m_2 \neq 0$. We find

$$\begin{aligned}
\xi_3 &= e^{i\sqrt{1+\beta^2}\phi^1}\left(\partial_{\theta_2} + \frac{i}{\sqrt{1+\beta^2}}\cot\theta_2\partial_{\phi_2} - \frac{i}{\sqrt{1+\beta^2}}\text{cosec}\ \theta_2\partial_\psi\right)\,,\\
\xi_4 &= \xi_3^*\ .
\end{aligned} \tag{D.15}$$

Let us now set $m_1 \neq 0, m_2 \neq 0$ and $m_3 = 0$. The (ψ, ϕ_1) and (ψ, ϕ_2) components together gives

$$\begin{aligned}
im_1\xi_\psi - \sin\theta_1\xi^{\theta_1} &= 0\\
im_2\xi_\psi - \sin\theta_2\xi^{\theta_2} &= 0\ .
\end{aligned} \tag{D.16}$$

Eliminating ξ_ψ from the above two equations, we find

$$\frac{\xi^{\theta_1}}{\xi^{\theta_2}} = \frac{m_1\sin\theta_2}{m_2\sin\theta_1}\ . \tag{D.17}$$

Since ξ^{θ_1} is independent of θ_1 and ξ^{θ_2} is independent of θ_2, the above equation can be met only if ξ^{θ_1} is proportional to $\sin\theta_2$ and vice versa. From $\partial_{\theta_1}\xi^{\theta_2} + \partial_{\theta_2}\xi^{\theta_1} = 0$ we find $\partial_{\theta_1}\partial_{\theta_2}\xi^{\theta_2} = 0$, indicating the proportionality constants must be zero. From the above discussion, we get $\xi^{\theta_1} = \xi^{\theta_2} = \xi_\psi = 0$. It is now easy to see from the (ϕ_1, ϕ_1) and (ϕ_2, ϕ_2) components of the killing equation that $\xi_{\phi_1} = \xi_{\phi_2} = 0$. And hence we don't have any killing vector for the above choice of m_1, m_2, m_3. In a similar manner, we can show hat we don't have any nontrivial solution to the killing equations when $m_1 \neq 0, m_3 \neq 0$ and $m_2 = 0$ as well as when $m_2 \neq 0, m_3 \neq 0$ and $m_1 = 0$.

In summary, the metric, eq. (5.9), has seven Killing vectors, given in eq. (5.14).

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E. General canonical form of charge vector in $\Gamma^{6,6}$

We start with a charge vector $\vec{Q} \in \Gamma^{2,2}$, where $\Gamma^{2,2} = \mathcal{H} \oplus \mathcal{H}$, is the 4-dimensional lattice made out of two 2-dimensional Hyperbolic lattices, $\mathcal{H}$. In components, $\vec{Q}$ takes the form,

$$\vec{Q} = (a, -b, c, d). \tag{E.1}$$

The lattice, $\Gamma^{2,2}$, is invariant under the action of $O(2,2,\mathbb{Z})$. We show that using an $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \in O(2,2,Z)$ the vector, $\vec{Q}$, can be brought to the form,

$$\vec{Q} = (gcd(\vec{Q}), \frac{\vec{Q}^2}{gcd(\vec{Q})}, 0, 0), \tag{E.2}$$

where,

$$gcd(\vec{Q}) = gcd(a, b, c, d), \tag{E.3}$$

and

$$\vec{Q}^2 = \vec{Q} \cdot \vec{Q}. \tag{E.4}$$

Note that the only non-vanishing components in eq. (E.2) lie in the first $\mathcal{H}$ sublattice.

It is useful for this purpose to represent $\vec{Q}$ as a 2×2 matrix,

$$Q = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}. \tag{E.5}$$

The first $\mathrm{SL}(2,\mathbb{Z})$, which we denote as $\mathrm{SL}(2,\mathbb{Z})_T$, acts on the left and performs row operations, while the second $\mathrm{SL}(2,\mathbb{Z})$, which we denote as $\mathrm{SL}(2,\mathbb{Z})_U$, acts on the right and carries out column operations. If $A \in \mathrm{SL}(2,\mathbb{Z})_T, B \in \mathrm{SL}(2,\mathbb{Z})_U$, then under their action,

$$Q \to AQB. \tag{E.6}$$

Note that $\vec{Q}^2 = det(Q)$. We will show that A, B can be found which bring Q to the form,

$$Q = \begin{pmatrix} gcd(\vec{Q}) & 0 \\ 0 & \frac{det(Q)}{gcd(\vec{Q})} \end{pmatrix} \tag{E.7}$$

This is equivalent to $\vec{Q}$ taking the form, eq. (E.2).

It is enough to prove this result for the case when $gcd(\vec{Q}) = 1$, in which case, eq. (E.7) becomes,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & det(Q) \end{pmatrix}. \tag{E.8}$$

The more general result, eq. (E.7), then follows, by considering the vector, $\frac{1}{gcd(\vec{Q})}\vec{Q}$, which has unit value for its gcd. In the discussion below we will sometimes use to the notation,

$$gcd(Q) \equiv gcd(\vec{Q}) = gcd(a, b, c, d). \tag{E.9}$$

The proof is as follows. Given any 2 integers, Euclid gives us an algorithm to arrive at their gcd in the following fashion. Subtract the smaller of the 2 numbers from the larger and then if the result is still larger than the smaller number continue this operation till the result becomes otherwise. Then start subtracting the new smaller number from the new larger number and continue this set of steps till one of the numbers becomes zero at which point the other number is the gcd. If the two integers are a, c, the two elements of the first column of matrix, Q, eq. (E.5), then this sequence of operations can be implemented by an element of $Sl(2,\mathbb{Z})_T$ which acts on the left and carries out row operations. The resulting form of Q is,

$$Q = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}, \tag{E.10}$$

where $a' = gcd(a,c)$. Note that $gcd(Q)$ is preserved by this operation. Since $gcd(Q) = 1$, to begin with, we learn that,

$$gcd(a', b', d') = 1. \tag{E.11}$$

Now we come to the crucial step. Let $\{p_1, \cdots p_r\}$, be the set of distinct primes which divide d' but do not divide b'. Let $m = \Pi p_i$, be the product of all these primes. One can show that the two numbers, d', and, $a'm + b'$, are coprime. Let p' be a prime that divides d', then if it does not divide b' it must divide m (by construction) and thus cannot divide $a'm+b'$. If on the other hand p' divides b', it cannot divide m (again by construction) and also it cannot divide a' (since eq. (E.11) is valid), and therefore p' cannot divide $a'm + b'$. Thus, we learn that $gcd(d', a'm + b') = 1$ and these two numbers are coprime.

We use this result to bring Q, eq. (E.10), to the form, eq. (E.8). First, an $\mathrm{SL}(2,\mathbb{Z})_U$ transformation can be carried out,

$$Q \to Q \begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a' & a'm + b' \\ 0 & d' \end{pmatrix}. \tag{E.12}$$

Since $gcd(a'm+b', d') = 1$, we can use Euclid's algorithm as in the discussion above to now find an $\mathrm{SL}(2,\mathbb{Z})_U$ transformation which bring Q to the form,

$$Q = \begin{pmatrix} a'' & 1 \\ c'' & d'' \end{pmatrix}. \tag{E.13}$$

Next, further $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ tranformations can be carried out to subtract the second column from the first a'' times, and the first row from the second d'' times. This followed by a row- column interchange operation gives Q in the form, $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. Since these operations preserve the determinant, we learn that $u = det(Q)$, leading to eq. (E.8).

We end by making a few points. First, note that this argument holds for space-like, time-like and null charge vectors, Q. Second, it follows from our analysis that there are two independent invariants for $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. These are $det(Q)$ and $gcd(Q)$. Of these $det(Q)$ is an invariant of the continuous group, while $gcd(Q)$ is a discrete invariant. Third, if instead of $\Gamma^{2,2}$ we start with a lattice which is the direct sum of more than two copies of $\mathcal{H}$, a similar argument can be used sequentially on the first two $\mathcal{H}$ sublattices, then the first and third $\mathcal{H}$ sublattices etc, to finally bring the charge vector to the form,

$$\vec{Q} = (gcd(\vec{Q}), \frac{\vec{Q}^2}{gcd(\vec{Q})}, 0, 0, \cdots, 0, 0). \tag{E.14}$$

In particular this is true for $\Gamma^{6,6}$ which consists of six copies of $\mathcal{H}$. Finally, if there are two charge vectors, $\vec{Q}_e, \vec{Q}_m$, then the above argument can be used to put one of them, say $\vec{Q}_e$, in the form, eq. (E.14). Further transformations which act trivially on the first Hyperbolic sublattice will keep $\vec{Q}_e$ invariant. Using these transformations $\vec{Q}_m$ can now be brought to the form,

$$\vec{Q}_m = (\alpha, \beta, \gamma, \delta, 0, 0 \cdots, 0, 0), \tag{E.15}$$

so that only the components in the first two Hyperbolic sublattices are non-vanishing. These results apply in general to the cases when $\vec{Q}_e^2, \vec{Q}_m^2$, have space-like, time-like or null norms.

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